Problem 1 *Game Theory.* Solve the followng games for the *optimal strategy* and *value* of the game.

(a) $A = \begin{pmatrix} 10 & 9 & 0 & 3 \\ 7 & 8 & 1 & 3 \\ 5 & 3 & 2 & 4 \\ 5 & 2 & 0 & 2 \\ 0 & 3 & 0 & 1 \end{pmatrix}$

Solution. The key to remember is that player I (choosing rows) wants to maximize while player II wants to minimize. There will be a non-mixing optimal strategy if there exists an entry that is both the max of its column and the min of its row. The only entry that satifies this is the 2 in position row 2, col 2. The value of the game is 2 as well.

You could also solve this with row / column reduction.

(b) $B = \begin{pmatrix} 5 & 2 \\ 0 & 5 \end{pmatrix}$

Solution. Here there is no maxmin entry, so the optimal strategy is to mix. Setting the column mixes equal gives $5\lambda = 2\lambda + 5(1-\lambda)$, or $\lambda = 5/8$. The row mixing equation is exactly the same, and so player I will randomize with proportions $(5/8, 3/8)$, player II with proportions $(3/8, 5/8)$ and the value is $5 * 5/8 = \frac{25}{8}$.

Problem 2 *Bradley-Terry.* Three citites, comprised of independent and identically distributed citizens, with respective populations n_1,n_2, and n_3 each have a champion (the best of their city). What is the probability that city 1 has the champion of champions and city 2 has the champion of the remaining two?

Solution. A key point of the cities model is champions are selected once and then compared between cities, and not redrawn for each city matchup. A concrete example is if the competiton was for heaviest citizen. Thus the event that we are looking for is that the heaviest person is in city 1 and the heaviest person among cities 2 and 3 is in city 2. Since each citizen is identical, and so all ordering are equally likely, this is just the event that we randomly select a citizen in city 1 from all three cities and one in city 2 from the remaining 2, or $\frac{n_1}{n_1+n_2+n_3}\frac{n_2}{n_2+n_3}$.

Problem 3 *Point Spreads.* Use Hal Stern's approximation $p_w = .5 + .03p$ for this one.

Team A is a 3 point favorite to beat team B in the first game and a 3 point underdog to beat team C in the second game.

(a) What is the probability that team A wins both games?

Solution. The matchups are independent and have win probabilities .59 and .41, so $Pr(A \text{ wins both}) = .59 * .41 = 0.2419$.

(b) Compare this to the probability of getting 2 Heads in 2 flips of a coin.

Solution. $Pr(HH) = .5 * .5 = 0.25$.

(c) What is the probability that team B beats team C? You may use Bradley-Terry.

Solution.

$$Pr(B > C) = \frac{Pr(B > A)Pr(A > C)}{Pr(B > A)Pr(A > C) + Pr(A < B)Pr(A < C)} = \frac{(1-.59)(.41)}{(1-.59)(.41) + (.59)(1-.41)} = 0.326$$

Or about a 1/3 chance.

(d) Given this value, what would be the corresponding point spread in a game between B and C?

Solution. Plugging this into Hal Stern's approximation gives, $.326 = .5 + .03p$ or $p = -5.8$. Team B is about a 6 point underdog to team C.

Problem 4 *Martingales.* Let $p(t)$, $0 \leq t \leq 1$, be the probability that team A beats team B, given the information $X(t')$, $0 \leq t' \leq t$ up to time t. Assume $p(t)$ is continuous, $p(0) = .5$ (teams equally matched), and $p(1) = 0$ or 1 (the game is over at time 1).

What is the expected number of crossings from one side to the other of the interval $[.3, .7]$ by $p(t)$?

Solution 1. We write out the expected number of crossing explcitly as an infinite sum.

Luckily .3 and .7 are symmetric about the starting point of .5, and $p(t)$ will certainly hit at least one of these values before being absorbed at 0 or 1. Starting from .3 or .7, the probability of one crossing to the other side is $\frac{.3}{.7} = 3/7$. Also the probability of being absorbed before completing the crossing is just $1 - 3/7 = 4/7$. We use these to write an infinite sum as follows, letting C count the number of crossings then the probability of exactly i crossings is $Pr(C = i) = (3/7)^i(4/7)$ and

$$E[C] = \sum_{i=1}^{\infty} iPr(C = i) = \sum_{i=1}^{\infty} i(3/7)^i(4/7) = 3/4$$

Solution 2. Use recurrence. Again we assume the martingale $p(t)$ is at either .7 or .3. What are the different possible paths? It can either be absorbed or hit the opposite boundary. We calculated these probabilities in solution 1 as 4/7 and 3/7. Only in the second path do we get a crossing and the recurrence continues. The recurrence equation then becomes

$$E[C] = (4/7) * 0 + (3/7) * (1 + E[C])$$

And solving for $E[C]$ gives $E[C] = 3/4$.

Note there are many other ways of decomposing the paths. As long as you keep the probabilities straight, you'll get 3/4.

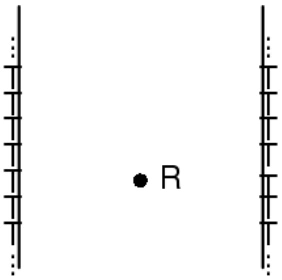
Problem 5 *Tacklers.* A runner is trapped half way between two infinitely long walls of tacklers as shown.

The walls are 50 yards apart.

(a) Where should the runner run to run as far as he can?

Solution. Since the runner is half way between the two walls, running straight north will allow him to run as far north as he is from either wall. Tacklers from either wall can run straight east or west, and one will intersect his path at exactly this distance.

If the runner runs an any angle other than due north, he will have to cover $1/cos(\theta)$ times as much distance to get to the same point as running straight north. It is not hard to calculate that the same strategy by all the tacklers (running straight east or west to the center of the field) will result in the runner being tackled at a $\frac{cos(\theta)^2}{1+sin(\theta)}$ proportion of the distance he can get by running straight north. Since both *sin* and *cos* are in absolute value less than 1, this proportion is always less than 1 as well, and he can do better by running north.



Any other path (such as a curve) will contain some amount of these angle routes, possibly an infinite amount, and so will be suboptimal to the straight north route.

(b) How far can he run?

Solution. The walls are 50 yards apart, and so his distance from either wall is 25 yards.

(c) Describe as best as possible the set of all points the runner can reach before the tacklers.

Solution. We showed in the first HW that with one infinitely long wall, the set of all possible points is a parabola, so with 2 walls we expect an intersection of two parabolas. The parabolas will intersect at two points, one 25 yards north of the runner, and the other 25 yards south. The directrix for each parabola is the line of tacklers.