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COMMENTARIES

Commentaries are informative essays dealing with viewpoints of statistical practice, statistical education, and other topics considered to be of general interest to the broad readership of *The American Statistician*. Commentaries are similar in spirit to Letters to the Editor, but they involve longer discussions of background, issues, and perspectives. All commentaries will be refereed for their merit and compatibility with these criteria.

Do Longer Games Favor the Stronger Player?

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1. INTRODUCTION

It is an article of faith that longer games favor the stronger player. Longer games offer a fairer test of skill, and they offer more evidence.

Certainly no statistician would turn down extra samples in a statistical test. More information does not hurt, because it can always be ignored. Although the argument about statistical tests is true, I argue that, as far as games go, the conclusion is generally untrue. And untrue not because of pathological counterexamples, but untrue even for simple games.

2. THE GENERAL GAME

We investigate a game of n periods between player A and player B. The first period is considered to be the basic game, and the n period game is the accumulation of scores in the basic game. Player A receives score X_i and B receives score Y_i in the ith period. Since the periods are of equal length, we shall assume that $X_1, X_2, \ldots$ are iid according to $F(x)$ and $Y_1, Y_2, \ldots,$ are iid according to $G(y)$. (We assume $\{X_i\}$ and $\{Y_i\}$ to be independent, but allowing dependence of X_i and Y_i does not affect the results.) Finally, let

$$P_n = \Pr\left\{\sum_{i=1}^{n} X_i > \sum_{i=1}^{n} Y_i\right\} \tag{1}$$

be the probability that player A wins the game (outscores his opponent). Fixing the distributions F and G determines the game.

3. ODD BEHAVIOR

First, consider an example where all is well. Let X_i and Y_i be independent standard normal with means μ_1 and μ_2, respectively, with $\mu_1 > \mu_2$. Then $P_1 > \frac{1}{2}$, and the probability P_n of winning the n-period game is monotonically increasing to 1. This is how we expect P_n to behave.

To be perverse we ask whether there exist scoring distributions $F(x)$ and $G(y)$ such that $P_1 > \frac{1}{2}$ and P_n is not monotonically increasing. An extreme example in which $P_1 = (\sqrt{5} - 1)/2 = .618$ and $P_2 = 1 - P_1 = .382$ is given. Thus playing the game for twice as long causes players A and B to switch roles as favorites at .62/.38 odds. The distributions for this example are

$$\begin{aligned} X &= 3, \ \alpha \\ &= 0, \ 1 - \alpha, \end{aligned}$$

and $Y \equiv 2$, where α will be chosen appropriately as follows. Clearly, $P_1 = \alpha$. In addition, player A wins a game of length 2 only if $X_1 = 3$ and $X_2 = 3$, which occurs with probability $P_2 = \alpha^2$. To obtain the example (in which $P_2 = 1 - P_1$) we solve $\alpha^2 = 1 - \alpha$, the equation for the golden ratio.

To see how bad it can get for player A we prove the following.

Lemma 1. For any P_1 and for any n period game,

$$1 - (1 - P_1)^n \geq P_n \geq P_1^n, \tag{2}$$

and there exist game distributions achieving these bounds.

Proof. Let $Z_i = X_i - Y_i$. Clearly, $Z_1, Z_2, \ldots$ are iid. Then, for any distribution on Z,

$$\begin{aligned} P_n &= \Pr\left\{\sum_{i=1}^{n} Z_i > 0\right\} \\ &\geq \Pr\{Z_i > 0, \ i = 1, 2, \ldots, n\} = P_1^n. \end{aligned}$$

Finally, the bound $P_n = P_1^n$ is achieved by the distribution

$$\begin{aligned} X &= n + 1, \quad P_1 \\ &= 0, \qquad 1 - P_1 \\ Y &\equiv n. \end{aligned} \tag{3}$$

The upper bound follows by reversing teams A and B in the argument.

Consequently, we could have player A win a game of length 1 with probability .99 but win a game of length 500 with probability .007. Player A slowly and steadily outscores B until a disastrous 0 occurs. Of course, in this example A's probability of winning will get worse as $n \to \infty$.

We now ask whether P_n can dip, rise, dip, etcetera, for-

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ever. Indeed it can, dropping from near 1 to near 0, rising back to near 1, and so on. The length n of the contest determines the "stronger" player. We have the following.

Lemma 2. There exist distributions $F(x)$, $G(y)$ and a subsequence of times $n_1, n_2, \ldots$, such that player A's win probability P_n satisfies

$$P_{n_i} \to 1, \qquad i = 1, 3, 5, \ldots, \tag{4}$$

and

$$P_{n_i} \to 0, \qquad i = 2, 4, 6, \ldots. \tag{5}$$

Proof. The idea of the proof is straightforward and the details (omitted here) can be provided in a number of unrewarding ways. We wish to find a distribution on $Z = X - Y$ that has increasingly spread-out mass points of alternating sign and appropriately decreasing probabilities. We want the sign of the sum $\Sigma_1^n Z_i$ to be determined by the largest term in the sum. Let $0 < a_1 < a_2 < \cdots$, and let $\Pr\{Z = (-1)^{k+1}a_k\} = \alpha_k$ $(k = 1, 2, \ldots)$. Let $N_k^{(n)}$ = the number of times $Z_i = (-1)^{k+1}a_k$ occurs in the sequence $Z_1, Z_2, \ldots, Z_n$. Let $S_n = \Sigma_{i=1}^n Z_i$.

Then

$$\sum_{i=1}^{n} Z_i = \sum_{k=1}^{\infty} N_k^{(n)} a_k (-1)^{k+1}. \tag{6}$$

By the law of large numbers

$$N_k^{(n)} = n\alpha_k + o(n\alpha_k), \tag{7}$$

where $o(n\alpha_k)/n\alpha_k \to 0$ with probability 1. We choose values a_k, probabilities α_k, and reversal times n_k to satisfy

$$n_k \alpha_k \to \infty, \tag{8}$$

$$n_k \sum_{k+1}^{\infty} \alpha_i \to 0, \tag{9}$$

and

$$a_k > n_k a_{k-1}. \tag{10}$$

Then, with probability tending to 1 as $k \to \infty$, Condition (8) implies that $N_k^{(n_k)} > 1$ with probability nearly 1, (9) implies that no terms of size a_{k+1} or greater are in S_{n_k}, and (10) guarantees that the terms in S_{n_k} of magnitude a_{k-1} or less are outweighed by the (1 or more) terms of size a_k. Thus the terms of size a_k determine the sign of S_{n_k}. But the sign of the a_k term is $(-1)^{k+1}$. Consequently,

$$P_{n_k} = \Pr\{S_{n_k} > 0\} \to 1, \qquad k = 1, 3, 5, \ldots,$$

$$P_{n_k} \to 0, \qquad k = 2, 4, 6, \ldots. \tag{11}$$

The choices

$$\alpha_k = 2^{-k^2} \Big/ \left(\sum_{1}^{\infty} 2^{-i^2}\right),$$

$$n_k = 2^{k^2+k}, \qquad a_k = 2^{k^3+k^2}, \tag{12}$$

satisfy conditions (8), (9), and (10), and lead to the desired example.

4. OPEN PROBLEM

Another kind of anomalous behavior that we have not pursued is the possibility of games such that $P_n > ½$, n odd, and $P_n < ½$, n even. Thus player A is favored only in games of odd length.

5. WHO IS THE STRONGER PLAYER?

It is tempting to say that player A is stronger if $P_1 > ½$. But then A may not be the favorite in long games. Alternatively, we might say A is stronger if $E(X - Y) > 0$, for then $P_n \to 1$. But now we have no guarantee of A's superiority in short games. Finally, if $E(X - Y)$ is not defined, we have examples where P_n oscillates between 0 and 1 forever. Here there is no well-defined notion of the stronger player.

The whole point is that the idea of the stronger player is not well defined until the length of the contest is specified.

6. REAL SPORTS

Sports like basketball and hockey probably have reasonably monotonic growth of P_n. Both consist of an accumulation of small equal-sized scores. Lindsey (1961) discussed the distribution of scores in baseball. Again, monotonicity of P_n seems to hold. In football we begin to have doubts. A team with a high-risk offense (passes and option plays) may be the favorite for half an hour but subject to disaster and loss in an hour contest. Similarly, in golf there are many long-hitting low-scoring golfers (like S. Ballesteros) who dominate short contests but are subject to infrequent catastrophic high scores that kick in during 72-hole matches.

Finally, in table stakes poker we observe that P_n is far from monotonic. The loosest player at the table plays more hands and thus wins more hands. He accumulates a series of antes and small pots until he comes up against the inevitable good hand, thereby losing more than his previous gains. A snapshot after half an hour of play will probably show him ahead, but in the long run he will be a loser.

Perhaps irregularity of the probability of winning is more common than believed.

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REFERENCE

Lindsey, G. R. (1961), "The Progress of the Score During a Baseball Game," *Journal of the American Statistical Association*, 66, 703–728.