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Problem Set #1
BioE 326B/Rad 226B

1. Maximizing image contrast
2. Thermal polarization
3. Eigenoperators

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Maximizing contrast (from de Graaf, problem 4.11)

a. For tissues A and B with $T_{2,A} = T_{2,B}$ but $T_{1,A} \neq T_{1,B}$, calculate the recovery delay TR that leads to the largest signal difference in a standard gradient-echo sequence. Assume a 90° excitation pulse and complete removal of transverse magnetization prior to excitation.

b. For tissues with $T_{1,A} = T_{1,B}$ but $T_{2,A} \neq T_{2,B}$, calculate the echo time TE that leads to the largest signal difference in a standard gradient-echo sequence. Assume a 90° excitation pulse and complete removal of transverse magnetization prior to excitation

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Thermal polarization (from de Graaf, problem 1.1)

Consider a spin system with two possible states having an energy difference of ΔE, where n_α and n_β are the number of spins in the lower and upper energy states respectively. From the Boltzmann distribution,

$$\left(\frac{n_\alpha}{n_\beta}\right) = e^{\Delta E/kT} = e^{\hbar\omega/kT}.$$

The NMR high temperature yields the following approximation:

$$\left(\frac{n_\alpha}{n_\beta}\right) = 1 + \left(\frac{\hbar\omega}{kT}\right).$$

Consider a 2 liter water-filled sphere (T=298.15K) placed inside a 3.0 T MR magnet.

a. Calculate the net excess of proton spins in the low-energy state (hint: water density = 1.00 g/mL and Avogadro constant = 6.02214×10^{23} mol–1).

b. Calculate the error made by ignoring the higher order terms in the high temperature approximation for T = 298.15 K, 4.0 K and 0.01 K.

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Eigenoperators

Just as operators have associated eigenkets, superoperators have associated eigenoperators.

Let $\hat{\hat{A}}$ be a commutation superoperator.

then, if $\hat{\hat{A}}\hat{B} = \lambda\hat{B}$ where λ is a scalar, $\hat{B}$ is an eigenoperator of $\hat{\hat{A}}$ with corresponding eigenvalue λ.

Consider the Hamiltonian $\hat{H}_0$ with eignenkets $|m\rangle$, i.e.

$$\hat{H}_0|m\rangle = \omega_m|m\rangle \text{ for } m = 1,\cdots,N$$

a. Show that the transition operators, defined as

$$\hat{T}_{nm} = |n\rangle\langle m|$$

are eigenoperators of $\hat{\hat{H}}_0$. What are their corresponding eigenvalues?

b. Let $\hat{H}_0 = -\omega_I\hat{I}_z - \omega_S\hat{S}_z$ and $\hat{I}_\pm = \hat{I}_x \pm i\hat{I}_y$, $\hat{S}_\pm = \hat{S}_x \pm i\hat{S}_y$

Show that $2\hat{I}_z\hat{S}_z$, $\hat{I}_\pm\hat{S}_\pm$, $\hat{I}_\pm\hat{S}_\mp$, $2\hat{I}_\pm\hat{S}_z$, $2\hat{I}_z\hat{S}_\pm$

are all eigenoperators of $\hat{\hat{H}}_0$, and complete the following table:

Eigenoperator	Eigenvalue	Corresponding transition operator(s)
$2\hat{I}_z\hat{S}_z$	0	$\hat{T}_{11} - \hat{T}_{22} - \hat{T}_{33} + \hat{T}_{44}$
$\hat{I}_+\hat{S}_+$		
$\hat{I}_-\hat{S}_-$		
$\hat{I}_+\hat{S}_-$		
$\hat{I}_-\hat{S}_+$		
$\hat{I}_+\hat{S}_z$		
$\hat{I}_-\hat{S}_z$		
$\hat{I}_z\hat{S}_+$		
$\hat{I}_z\hat{S}_-$		