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Constant d-Regular Expanders

Recall from last lecture the definitions of conductance, sparsity, and expansion:

Definition 8.1: Expansion The *expansion* $\rho(G)$ is the minimum edge cut between two sets divided by the size of the smaller set,

$$\rho(G) = \min_S \frac{C(S,\bar{S})}{\min(|S|,|\bar{S}|}$$

Definition 8.2: Sparsity The *sparsity* of a cut $sp(S,\bar{S})$ is

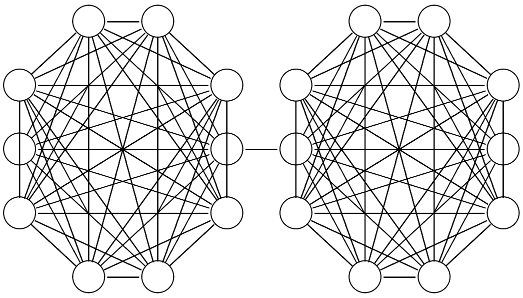
$$sp(S,\bar{S}) = \frac{C(S,\bar{S})}{\min(vol(S), vol(\bar{S}))},$$

where $vol(S) = \sum_{i \in S} d_i$.

Definition 8.3: Conductance The *conductance* of a graph $\phi(G)$ is

$$\phi(G) = \min_{s \subset V} sp(S,\bar{S})$$

Which is the graph with the lowest conductance/expansion? The dumbell graph (i.e. two n/2 cliques joined by a single edge):



Which has the highest expansion? The complete graph K_n.

Which d-regular graph has the highest expansion? This is not known.

Can we construct a d-regular graph with constant expansion? It actually took a long time to come up with a method of producing graphs with high expansion. The first proof of their existence came from a 70-80 page paper detailing a step-by-step procedure to improve the conductance of a regular graph through switching, with the maximum of this process having constant expansion.

We can show this much more easily, however, via the probabilistic method:

Theorem 8.1 *For all $d \geq 3$, there exists a constant $\alpha > 0$ such that with high probability a random d-regular graph has expansion $\rho \geq \alpha$.*

Proof: We will prove this theorem for d sufficiently large; the proof for $d \geq 3$ is similar. We can generate d regular graphs on n via the "configurational model": we split each vertex into d minivertices, and assign edges based on a random perfect matching of the mini-vertices. Note that a graph produced in this way may have multiple edges and self-loops.

To prove the theorem we need to show that for each set $S \subset V$, with $|S| = k \leq \frac{n}{2}$, the probability that $|C(S, \bar{S})| \leq \alpha k$ is sufficiently small.

Assume that there exists such an S of size k. For a given k, there are $\binom{n}{k}$ possible choices for S. For a given S, there are $\binom{dk}{\alpha k}\binom{dn-dk}{\alpha k}$ ways to choose the minivertices in S and minivertices in $\bar{S}$ involved in a cut of size αk.

Now we count the number of ways the $dk - \alpha k$ minivertices in S not involved in the cut attach to each other. Define $f(m)$ to be the number of matchings between m vertices,

$$f(m) = \frac{\binom{m}{2}\binom{m-2}{2}\cdots\binom{2}{2}}{(m/2)!} = \frac{m!}{(2^{m/2}(m/2)!}.$$

Then the probability of the event that only a certain αk of minivertices match outside of their proper subset is at most

$$\frac{f(dk-\alpha k)f(dn-dk-\alpha k)f(2\alpha k)}{f(dn)}$$

Therefore we have, the probability that there is a subset S with expansion at most α is

$$\sum_{k=1}^{n/2} \alpha k \binom{n}{k}\binom{dk}{\alpha k}\binom{dn-dk}{\alpha k}\frac{f(dk-\alpha k)f(2\alpha k)f(dn-dk-\alpha k)}{f(dn)}$$

In order to bound the above equation, we will use a version of Stirling's inequality:

$$\frac{n^k}{k} \leq \binom{n}{k} \leq \xi\left(\frac{ne}{k}\right)^k$$

Applying this to f shows that:

$$f(m) = c\frac{m^{m+(1/2)}e^{-m}}{2^{m/2}(m/2)^{(m/2)+(1/2)}e^{-(m/2)}} \leq \frac{cm^{m/2}}{e^{m/2}}.$$

By plugging in the above bounds we have:

$$\begin{aligned}
& \frac{f(dk-\alpha k)f(2\alpha k)f(dn-dk-\alpha k)}{f(dn)} \\
\leq\ & \sum_{k=1}^{(n/2)} \alpha k \left(\frac{ne}{k}\right)^{k} \left(\frac{edn}{dk}\right)^{(2\alpha k)} \frac{(dk-\alpha k)^{(dk-\alpha k/2)}(2\alpha k)^{(\alpha k)}(dn-dk-\alpha k)^{(dn-dk-\alpha k/2)}}{dn^{(dn/2)}} \\
\leq\ & \sum_{k=1}^{(n/2)} \alpha k \left(\frac{nec}{k}\right)^{(k+2\alpha k)} \left(\frac{dk}{dn}\right)^{((dk-\alpha k)/2)} \\
\leq\ & \sum_{k=1}^{(n/2)} \alpha k \left(\frac{k}{n}\right)^{((d-2)k-5\alpha k)/2} (ec)^{k+2\alpha k} \quad \text{for } \alpha \text{ sufficiently small} \\
\leq\ & \sum_{k=1}^{(n/2)} \left(\frac{k}{n}\right)^{((d-3)k)/2)} (ec)^{(2k)} \quad \text{for sufficiently large } d
\end{aligned}$$

Since $k/n \leq 1/2$, $(k/n)^{2c} \leq (ec)^{-2}$. So choosing d "sufficiently large", we can bound the above by:

$$\begin{aligned}
\leq\ & \sum_{k=1}^{(n/2)} \left(\frac{k}{n}\right)^{2k} \alpha k \\
=\ & \alpha\left(\frac{1}{n}\right)^{2} + 2\alpha\left(\frac{2}{n}\right)^{4} + \ldots + \left(\frac{(n/2)}{n}\right)^{(n/2)} \frac{n\alpha}{2} \\
\leq\ & \alpha\left(\frac{1}{n}\right)^{2} + 2\alpha\left(\frac{2}{n}\right)^{4} + \frac{n}{2}3\alpha\left(\frac{3}{n}\right)^{6} \\
\leq\ & \frac{1}{n}
\end{aligned}$$

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For more details, see Chapter 3 of Durrett's book.