MS&E 337 **Lecture #6 Notes**
Information Networks **Fall 2007**
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Spectrum of a graph

Let $G(V, E)$ be a simple undirected graph. Asume $|V| = n$ and $|E| = m$. The adjacency matrix $A_{n\times n}(G)$ is such that

$$A_{ij} = \begin{cases} 1 & : i \sim j \\ 0 & : \text{otherwise} \end{cases}$$

The Laplacian of G, $L_{n\times n}(G)$ is defined as

$$L_{ij}(G) = \begin{cases} d_i & : i = j \\ -1 & : i \sim j \\ 0 & : \text{otherwise} \end{cases}$$

The Laplacian is often defined in terms of A as $L = D - A$, where D is the diagonal matrix with $D_{ii} = d_i$. Alternatively, you can write $L(G) \sum_{i\sim j} L(i, j)$ where

$$L_(i, j) = \begin{array}{c|ccccccc} & 1 & \dots & i & \dots & j & \dots & n \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & \\ \vdots & 0 & 0 & 0 & 0 & 0 & 0 & \\ i & 0 & 0 & 1 & 0 & -1 & 0 & \\ \vdots & 0 & 0 & 0 & 0 & 0 & 0 & \\ j & 0 & 0 & -1 & 0 & 1 & 0 & \\ \vdots & 0 & 0 & 0 & 0 & 0 & 0 & \\ n & 0 & 0 & 0 & 0 & 0 & 0 & \end{array}$$

Why is this positive semi-definite? Because $x^T L x = \sum_{i\sim j}(x_i - x_j)^2$ for all $i \sim j$.

Let $0 \le \lambda_1 \le \ldots \le \lambda_n$ be the eigenvalues of L. The first eigenvector $v_1 = \mathbf{1}$ and $\lambda_1 = 0$.

Proposition 6.1 *G is connected iff* $\lambda_2 > 0$.

Proof:

Let V_2 be the second eigenvector if $\sum_{i \sim j}(v_2(i) - v_2(j))^2 = 0$, then $v_2(i) = v_2(j)$ for i and j in the same connected component. Therefore, if G is connected and $\lambda_2 = 0$, then $v_2 = c\mathbf{1}$, for some constant c. This contradicts $v_2 \perp v_1$.

If G can be partitioned into S_1 and S_2 suth that there is no edge between S_1 and S_2, then $v_2 = \alpha\mathbf{1}S_1 + \beta\mathbf{1}S_2$ such that

$$\begin{aligned} \alpha|S_1| + \beta|S_2| &= 0 \\ \alpha^2|S_1| + \beta^2|S_2| &= 1 \end{aligned}$$

would be a vector such that $v_2 \perp v_1$ and $v_2^T A v_2 = 0$.

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Corollary 6.1 *The number of connected components is equal to the multiplicity of the* 0 *eigenvalue.*

Remark: The eigenvectors and eigenvalues of a matrix are relatively resilient to random perturbations.

Let G be a connected simple undirected graph. A simple random walk on the vertices of G is as follows:

- Start with a particle at one of the vertices in the graph.
- At every time t it moves from its current place v to one of the neighbors of v chosen uniformly at random.

Let $v_0, v_1, \ldots, v_t$ be the sequence of vertices visited by the particle. If $P_t(i) = \Pr(v = i, t)$ then $P_t = (M^T)^t P_0$ where $M = D^{-1}A$.