1. Preferential Attachment

The preferential attachment model was introduced by Barabási and Albert to explain the power-law degree distribution in complex real-world networks. Here is their model, stated more formally.

Fix $m > 0$ constant. Let $G_1, G_2, \ldots, G_t, \ldots$ be a sequence of graphs such that G_t is obtained from G_{t-1} by adding v_t and m edges to $\{v_1, v_2, \ldots, v_{t-1}\}$ such that the probability of adding a link from v_t to v_i is proportional to the degree of v_i. Specifically, we define intermediate graphs $\Gamma_{t,1}, \Gamma_{t,2}, \ldots, \Gamma_{t,m}$ such that $\Gamma_{t,j}$ is obtained from $\Gamma_{t,j-1}$ by adding a link between edge v_t and v_l with probability $deg(v_l)/2(m(t-1)+j-1)$, and $G_t = \Gamma_{t-1,m} = \Gamma_{t,0}$. We initialize the process by setting G_1 to be a singleton with m loops.

Barabási and Albert ran Monte Carlo simulations of the preferential attachment process and estimated a power-law exponent of $\alpha = 2.9 \pm .1$. This observation led to the following heuristic "mean-field" argument for the true exponent to be $\alpha = 3$:

1. Let $d_i(t)$ be the degree of node i at time t:

   $$d_i(t) = deg(v_i), v_i \in G_t.$$

   If we think of this as a continuous process and take the derivative, we get:

   $$\frac{\delta d_i(t)}{\delta t} = \frac{m d_i(t)}{\sum_{j=1}^{n} d_j(t)} = \frac{d_i(t)}{2t} \tag{1}$$

2. Let t_i be the time that node i arrives. We can solve (1) to get:

   $$d_i(t) = m\sqrt{\frac{t}{t_i}} \tag{2}$$

3. Assuming that t_i is distributed uniformly on $[0, t]$, then

   $$\Pr\left(d_i(t) > k\right) = \Pr\left(m\sqrt{t} > k\sqrt{t_i}\right) = \Pr\left(t_i < \frac{m^2 t}{k^2} = \frac{m^2}{k^2}\right) \tag{3}$$

4. Differentiating (3) gives

   $$\Pr(d_i(t) = k) = \frac{2m^2}{k^3} \tag{4}$$

This gives a "back of the envelope" justification for the $\alpha = 3$ power-law exponent. We can prove this more rigorously.

Let $Z(k,t)$ be the random variable indicating the number of vertices of degree k at time t. Let $N(k,t) = \mathrm{E}(Z(k,t))$. Then

$$N(k,t) - N(k,t-1) = \frac{m(k-1)N(k-1,t)}{2m(t-1)} - \frac{mkN(k,t-1)}{2m(t-1)} + 1_{m=k} + \epsilon(k,t), \qquad (5)$$

where $\epsilon(k,t)$ accounts for the possibility of multiple edges and can be bounded by

$$|\epsilon(k,t)| = O\left(\sum_{i=2}^{m} \frac{(k-i)^i N_{k-i}(t)}{(mt)^i}\right) = O\left(\frac{k}{t}\right) = O\left(t^{-1/2}\right)$$

Solving the recurrence relationship (5) can be done by algebraic manipulations (see Durret's Random Graph Dynamics, pp 92-93). For us, it is slightly easier. You can verify that

$$N(k,t) = \frac{(t-1)m(m+1)}{k(k+1)(k+2)} + O(t^{-1/2}),$$

by verifying it at $k = m$ and in the recurrence relation.

All that is left is to show that $Z(k,t)$ is concentrated around its expectation. To prove this, we define a martingale process $Z_i(k,t) = \mathrm{E}(Z(k,t)|Y_1, Y_2, \ldots, Y_{i-1})$, where the Y_i's are the random choices made at time i. We can then evoke the Azuma-Hoeffding inequality:

Lemma 2.1 *(Azuma-Hoeffding inequality). Let* $(X_t)_{t=0}^n$ *be a martingale with* $|X_{t+1} - X_t| \leq c$ *for* $t = 0, \ldots, n-1$. *Then*

$$\Pr(|X_n - X_0| \geq x) \leq \exp\left(-\frac{x^2}{2c^2 n}\right).$$

If we note that $|Z_i(k,t) - Z_{i-1}(k,t)| \leq 2m$, then we are done.

2. Connections to Pólya Urn Scheme

We give an equivalent description of the preferential attachment process as a combination of several Pólya urn processes. The Pólya urn model is proposed and analyzed much earlier in the beautiful work of Pólya and Eggenberger in the early twentieth century.

Pólya urn scheme

In early twentieth century, Pólya proposed and analyzed the following model known as the Pólya urn model. Suppose we have an urn with r red balls and b blue balls. At each step i, we pick a ball uniformly at random from the urn and replace it with two balls of the same color.

Pólya showed that this model is equivalent to another process as follows. Choose a parameter (like "strength" or "attractiveness") p, and at each step, *independently* of the decision in previous steps, toss a coin with bias p and put a blue or red ball in the urn depending on the outcome. Pólya specified the distribution of p for which this mimics the urn model and showed that it is a β-distribution with appropriate parameters.

For a complete treatment of Pólya urn scheme we need to define exchangeability of random variables and cover de Finnetti's Theorem. You can find those in Durrett's book Probability: Theory and Examples. Here we only sketch the main idea:

Let

$$X_t = \begin{cases} 1 & : \text{ ball chosen at time } t \text{ is blue} \\ 0 & : \text{ otherwise} \end{cases}$$

Then

$$\Pr(X_1 = 1, X_2 = 0, X_3 = 1) = \frac{b}{r+b}\frac{r}{r+b+1}\frac{b+1}{r+b+2} = \Pr(\text{“rbb”’}) = \Pr(\text{“bbr”’}).$$

The property that the ordering of events doesn't affect their cumulative probability, only the number of events of each type, is known as *exchangeability.* The probability of seeing n_1 red balls and n_2 blue balls after $n = n_1 + n_2$ steps is

$$\begin{aligned} \Pr(\text{Number of red balls is } n_1) &= \frac{n!}{n_1!n_2!}\frac{r(r+1)\dots(r+n_1-1)b(b+1)\dots(b+n_2-1)}{(r+b)\dots(r+b+n-1)} \\ &= \frac{n!}{n_1!n_2!}\frac{(r+b-1)!}{(r-1)!(b-1)!}\frac{(r+n_1-1)!(b+n_2-1)}{(r+b+n-1)!} \end{aligned}$$

Suppose $n_1/n = x$, and taking the limit for each term,

$$\begin{aligned} &\lim_{n\to\infty} (r+n_1-1)!/n_1! \to n_1^{r-1} \\ &\lim_{n\to\infty} (b+n_2-1)!/n_2! \to n_2^{b-1} \\ &\lim_{n\to\infty} (r+b+n-1)!/n! \to n^{b+r-1}, \end{aligned}$$

the probability of seeing n_1 red balls converges to

$$\lim_{n\to\infty} \Pr(\text{Number of red balls is } n_1) \to \frac{(r+b-1)!}{(r-1)!(b-1)!}\frac{1}{n}x^{r-1}(1-x)^{b-1},$$

which is a random variable with distribution $\beta(r, b)$.

Preferential attachment revisited

It is not hard to see that there is a close connection between the the preferential attachment model of Barabási and Albert and the Pólya urn model in the following sense: every new connection that a vertex gains can be represented by a new ball added in the urn corresponding to that vertex. We use this idea to give an equivalent description of the scale-free graph which is easier to analyze. We will see throughout the paper the properties of this description that make it useful for understanding the graph.

To derive this representation, let us consider first a two urn model, with the number of balls in one urn representing the degree of a particular vertex k, and the number of balls in the other representing the sum of the degrees of the vertices $1, \ldots, k-1$. We will start this process at the point when $n = k$ and k has connected to precisely m vertices in $\{1, \ldots, k-1\}$. Note that at this point, the urn representing the degree of k has m balls, while the other one has $(2k-3)m$ balls.

Taking into account that the two urns start with m and $(2k-3)m$ balls, respectively, we see that the evolution of the two bins is a Pólya urn with strengths ψ_k and $1 - \psi_k$, where

$$\psi_k \sim \beta\left(m, (2k-3)m\right). \tag{6}$$

Having the above insight, we construct an alternative description of the preferential attachment model. Let $\psi_1 = 1$, and for every $2 \leq k \leq n$, we take ψ_k to be distributed according to $\beta(m, (2k-3)m)$. For $1 \leq k \leq n$, we take

$$\phi_k = \psi_k \prod_{j=k+1}^{n} (1 - \psi_j).$$

It is easy to see that $\sum_{k=1}^{n} \phi_k = 1$. Let

$$l_k = \sum_{j=1}^{k} \phi_k.$$

For every $a \in [0,1]$, we define $\kappa(a) = \min\{k : l_k \geq a\}$. Let $\{U_{i,k}\}_{1 \leq i \leq m, 1 \leq k \leq n}$ be independent random variables, uniform on $[0,1]$. For $k > j$, we draw an edge between k and j if for some $1 \leq i \leq m$ we have

$$j = \kappa(U_{i,k} l_{k-1}). \tag{7}$$

Theorem 2.1 *The random graph described above has the same distribution as the n-vertex preferential attachment model with parameter m.*

The proof of the above Theorem follows from the theory of Pólya urns. For the details of the proof, you can see [BBCS07]. The main advantage of this new representation is that it contains much more independence and is therefore much simpler to analyze. For example you can see the analysis of joint degree distributions, the limiting distribution of subgraphs in balls of any given radius k around a random vertex in the preferential attachment graph in [BBCS07].