November 30, 2007
Prepared by Adam Guetz

Six degrees of separation

In the model of Watts and Strogatz, the so called "small world" model, we have a grid with a few random edges. The random edges are added with probability $\Pr(u,v) \propto \frac{1}{d(u,v)^\alpha}$ for some parameter α, where d is the grid distance, $d((i,j),(k,l)) = |i-k| + |j-l|$.

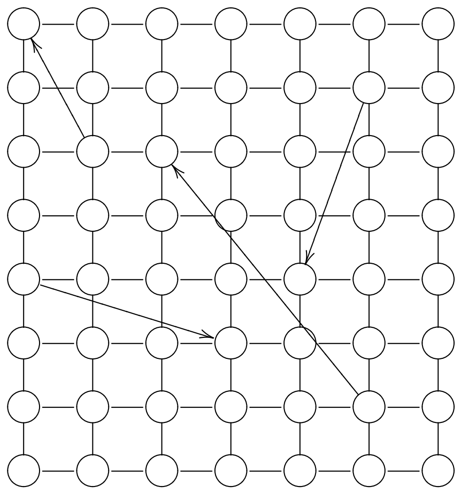

Figure 1: Model of Watts and Strogatz

Given a starting node v_1 and a target node v_k, a decentralized search algorithm finds a sequence $v_1, v_2, \ldots, v_k$. In each step, node v_i forwards the message to v_{i+1} having the following info:

- all short links
- coordinates of the destination
- coordinates and long range links of $v_1, v_2, \ldots, v_i$

A simpler model related to the Watts-Strogatz model is obtained by adding a random matching to a cycle (the Bollobás-Chung (BC) model).

Theorem 12.1 *The average length of the path created by ANY decentralied algorithm on the BC model is* $\Omega(n^{2/3})$.

Greedy Decentralized Search

Node s and t are chosen uniformly at random on the grid. Start at node t, at each step traverse the edge that ends closest to t.

Theorem 12.2 *When $r = 2$, the expected delivery time of greedy algorithm is $O(\log^2 n)$.*

Theorem 12.3 *For any value of $r \neq 2$, the expected dilevery time of any decentralized algorithm is $\Omega(n^\epsilon)$ for a constant $\epsilon = f(r)$.*

Proof:(Thm. 12.2) We can bound the sum of inverse squared distances as

$$\sum_v d^{-2}(u,v) \leq \sum_{j=1}^{2n-2} 4j\frac{1}{j} \leq 4\ln(2n)$$

We say the algorithm is in phase j iff the distance of the current node v from t is between 2^j and 2^{j+1}. Assume j is between $\log\log n < j < \log n$. Let B_j be the set of nodes of distance 2^j from t or less.

$$|B_j| \geq \sum_{i=1}^{2^j} i \geq 2^{2j-1}$$

The distance of node v from any node in B_j is at most $2^{j+1} + 2^j \leq 2^{j+2}$. The probability that v has a long range link to a node in v_j is at least

$$\frac{2^{2j-1}}{4\ln(2n)2^{2j+4}} = \frac{1}{128\ln(2n)}$$

Therefore, the expected number of steps in phase j is at most $128\ln(n)$, and the expected delivery time is $\leq \log n + \log n * 128\ln n = O(\log^2 n)$. ■