Lecture 5 (October 26, 2005)

prepared by Paul Constantine

1. Reminder: Expanders

Before we get to the main topic of this section, we first restate a couple of important definitions. The expansion of a graph $G(V, E)$ is defined as

$$\rho(G) = \min_{S \subset V} \frac{C(S, \bar{S})}{\min(|S|, |\bar{S}|)}$$

Also, the volume of a subset S is defined as

$$\mathrm{vol}(S) = \sum_{i \in S} d_i$$

where d_i is degree i. Then define the conductance of a graph is defined as

$$\phi(G) = \min_{S} \frac{C(S, \bar{S})}{\min(\mathrm{vol}(S), \mathrm{vol}(\bar{S}))}$$

From the definitions it follows that

$$\phi(G) \leq \rho(G) \leq \phi(G) d_{max}.$$

Now here's a variant of the theorem we discussed last lecture:

Theorem 5.1 *For all $d \geq 3$ and n sufficiently large, there exists an $\alpha > 0$ such that any random d-regular graph of size n has expansion α.*

2. Spectrum of a Graph

The adjacency matrix of a graph of with n vertices is an $n \times n$ matrix where

$$A(i,j) = \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & \text{otherwise} \end{cases}$$

We will examine the eigenvalues and eigenvectors of this matrix. Another useful matrix representation of a graph is called the *Laplacian*. The Laplacian of a graph $L(G)$ is defined as

$$L_G(i,j) = \begin{cases} d_i & \text{if } i = j \\ -1 & \text{if } i \text{ is adjacent to } j \\ 0 & \text{otherwise} \end{cases}$$

We can also define the Laplacian of a single edge as

$$L_{G(u,v)}(i,j) = \begin{cases} 1 & \text{if } i = j \in \{u,v\} \\ -1 & \text{if } i = u \text{ and } j = v \text{ or vice versa} \\ 0 & \text{otherwise} \end{cases}$$

It follows then that

$$L_G = \sum_{(u,v)\in E(G)} L_{G(u,v)}$$

It is easy to see that L_G is positive semi-definite since

$$x^T L_G(u,v) x = (x_u - x_v)^2$$

which implies

$$x^T L_G x = \sum_{(i,j)\,\in\,E} (x_i - x_j)^2$$

Now let $0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ be the eigenvalues of L. Clearly $\lambda_1 = 0$ since an n-vector of 1's lies in the null space.

Claim 5.1 $\lambda_2 > 0 \Leftrightarrow G$ *is connected.*

Proof: By contradiction, assume G is connected and $\lambda_2 = 0$. Then there exists a non-zero x orthogonal to $e = \{1, \ldots, 1\}^T$, which implies $\sum_i x_i = 0$, such that $x^T L x = 0$, or equivalently

$$\sum_{(i,j)\,\in\,E} (x_i - x_j)^2 = 0.$$

This implies $x_i = x_j$ for all i adjacent to j. But by assumption G is connected, so $x_i = x_j$ for all i, j. This is a contradiction since the components of a constant vector cannot sum to zero unless that constant is zero. Now suppose G is disconnected; we will show that λ_2 must be zero. Partition the vertices of G into sets S and $\bar{S}$. If S and $\bar{S}$ are disconnected, then for an n-vector $x = \alpha \mathbf{1}_S + \beta \mathbf{1}_{\bar{S}}$ we have $x^T L x = 0$, and we can set α and β such that $\|x\| = 1$ and $x^T e$. Therefore $\lambda_2 = 0$. ■

Corollary 5.1 *The multiplicity of the zero eigenvalue is equal to the number of connected components in G.*

The values of λ_2 can take ranges from 0 to $\mathcal{O}(d_{max})$. If graph is disconnected then $\lambda_2 = 0$ and the further λ_2 is from 0, the better the connectedness of the graph.