# MS&E 337 Information Networks
# Fall 2007
# Homework #1
# Due Wed. 10/31

1. Suppose we run the Pólya urn scheme starting with an urn with $r$ red and $b$ blue balls. What is the probability that a ball chosen at step $k$ is red?

2. Let $X$ be the number of triangles in the $G(n, p)$ model. Set $p = c/n$ for constant $c > 0$.

   (a) Find $\lim_{n\to\infty} \mathrm{E}[X]$.

   (b) Find $\lim_{n\to\infty} \Pr[X = 0]$.
   Hint: use Janson's inequality:

   **Theorem 1** *(Janson's inequality) Let $A_i$, $i \in I$, be subsets of $\Omega$, $I$ a finite index set. Let $B_i$ be the event $A_i \in R$, where $R$ is a random subset of $\Omega$. Let $i \sim j$ denote $i \neq j$ and $A_i \cap A_j \neq \emptyset$. Define*

   $$\Delta = \sum_{i\sim j} Pr[B_i \wedge B_j],$$

   $$M = \prod_{i\in I} Pr[\bar{B}_i]$$

   $$\mu = E[X] = \sum_{i\in I} Pr[B_i].$$

   *Note that $\Delta$ is over all ordered pairs $i \sim j$, so $\Delta/2$ is the sum over unordered pairs. Assume $Pr[B_i] \leq \epsilon$ for $i \in I$. Then*

   $$M \leq Pr[\wedge_{i\in I} \bar{B}_i] \leq M \exp\left(\frac{1}{1-\epsilon}\frac{\Delta}{2}\right)$$

   (c) (Bonus) A graph $H(V, E)$ is *balanced* if the maximal value of $|E|/|V|$ among all subgraphs of $H$ is achieved for $H$ and *strictly balanced* if that ratio is only achieved for $H$. Let $H(V, E)$ be a strictly balanced graph with $a$ automorphisms.

   Show that for $p = cn^{-|V|/|E|}$, $c > 0$ constant,

   $$\lim_{n\to\infty} \Pr[G \in G(n,p) \text{ contains no copy of } H] = \exp(-c^{|E|}/a).$$

3. Consider the following model for a collection of wireless devices trying to form a network using line-of-sight communications in an urban environment. The model is based on a simple abstraction of the idea that there are $n$ streets running east-west, $n$ avenues running north-south, and wireless nodes can be placed at intersections of streets and avenues. Due to the line-of-sight constraints, any pair of these wireless nodes can communicate if and only if they are on the same street or the same avenue.

   More concretely, we have an $n \times n$ grid of equally spaced points in the plane (representing the intersections), and we have $k$ wireless nodes residing at $k$ of the points. Two nodes can communicate iff they belong to the same row or column of the grid. Base on this definition, we define the *communication graph* $G$ on the $k$ nodes, with an edge connecting two nodes if they communicate. Note that $G$ may not be connected.

   Problem: given the grid as above with $k$ wireless nodes already placed at points in it, find a polynomial time algorithm that makes $G$ connected by adding the minimum possible number of wireless nodes to points in the grid.

4. Look at the internet toplogy network snapshots available at:
   `http://topology.eecs.umich.edu/data.html`.

   Parse the graph and plot its degree sequence, connected components, etc. using David Gleich's MatlabBGL software:
   `http://www.stanford.edu/~dgleich/programs/matlab_bgl/`.

   What do you observe?