# CLP Dual Interpretations and Duality Applications

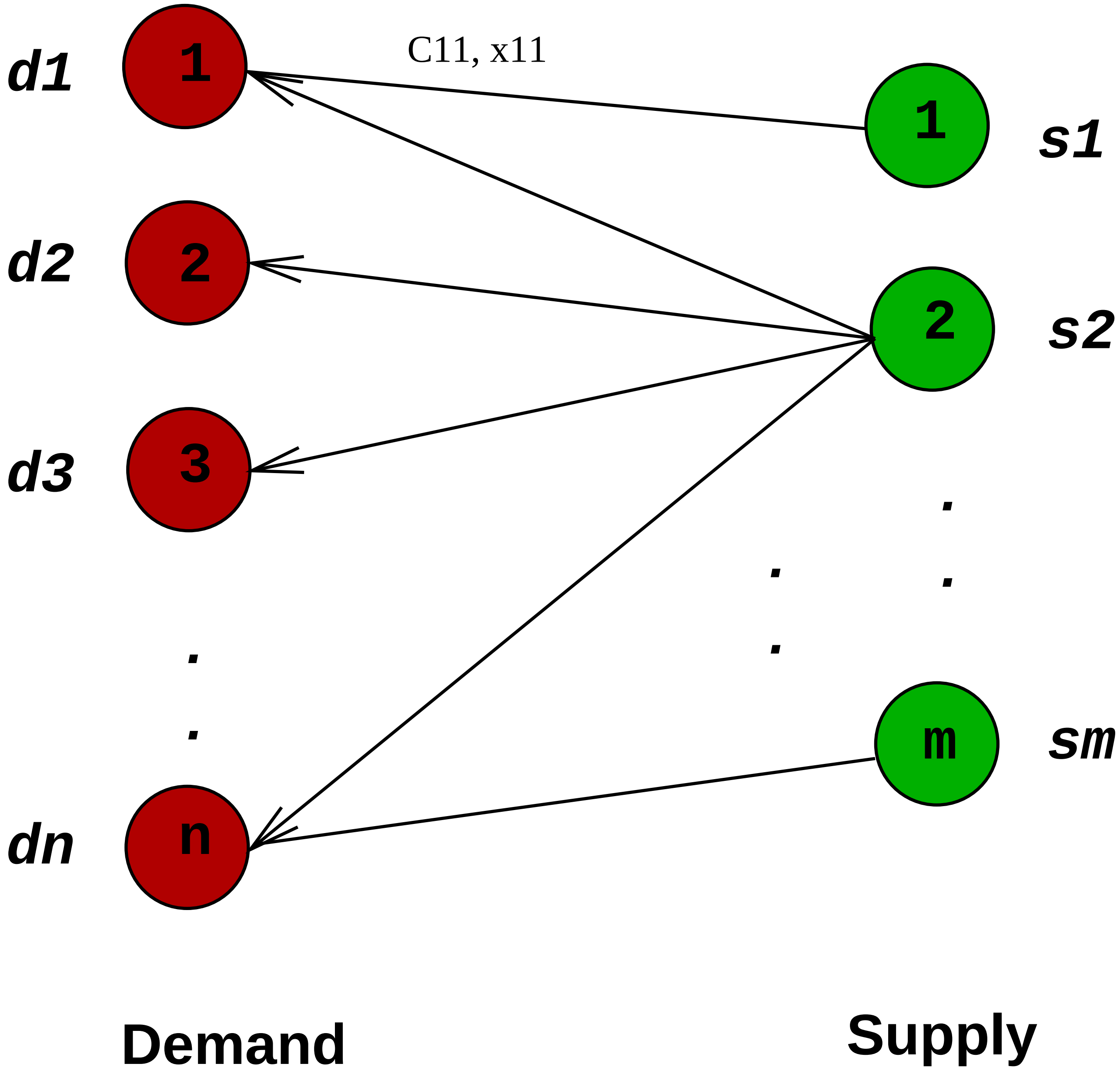
Yinyu Ye
Department of Management Science and Engineering
Stanford University
Stanford, CA 94305, U.S.A.

http://www.stanford.edu/~yyye

This week: Appendix B, Chapters 2.2, 2.6, 3.1-3.6, 6.3-6.4

## Recall Transportation Problem

$$
\begin{array}{lll}
\min & \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} & \\
\text{s.t.} & \sum_{j=1}^{n} x_{ij} & = s_i,\ \forall i = 1, ..., m \\
& \sum_{i=1}^{m} x_{ij} & = d_j,\ \forall j = 1, ..., n \\
& x_{ij} & \geq 0,\ \forall i, j.
\end{array}
$$


d1
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## Transportation Dual: Economic Interpretation

pricing

$$\begin{array}{lll} \max & \sum_{i=1}^{m} s_i u_i + \sum_{j=1}^{n} d_j v_j & \\ \text{s.t.} & u_i + v_j & \leq c_{ij}, \ \forall i, j. \end{array}$$

$u_i$: supply site unit price

$v_i$: demand site unit price

$u_i + v_j \leq c_{ij}$: competitiveness

## Algorithmic Applications: Optimal Value Function and Shadow Prices

$$z(\mathbf{b}) = \quad \text{minimize} \quad \mathbf{c}^T\mathbf{x}$$
$$\text{subject to} \quad A\mathbf{x} = \mathbf{b}, \ \mathbf{x} \geq \mathbf{0}.$$

Suppose a new right-hand-vector $\mathbf{b}^+$ such that

$$b_k^+ = b_k + \delta \quad \text{and} \quad b_i^+ = b_i, \ \forall i \neq k.$$

Then, the optimal dual solution $\mathbf{y}^*$ has a property

$$y_k^* = (z(\mathbf{b}^+) - z(\mathbf{b}))/\delta$$

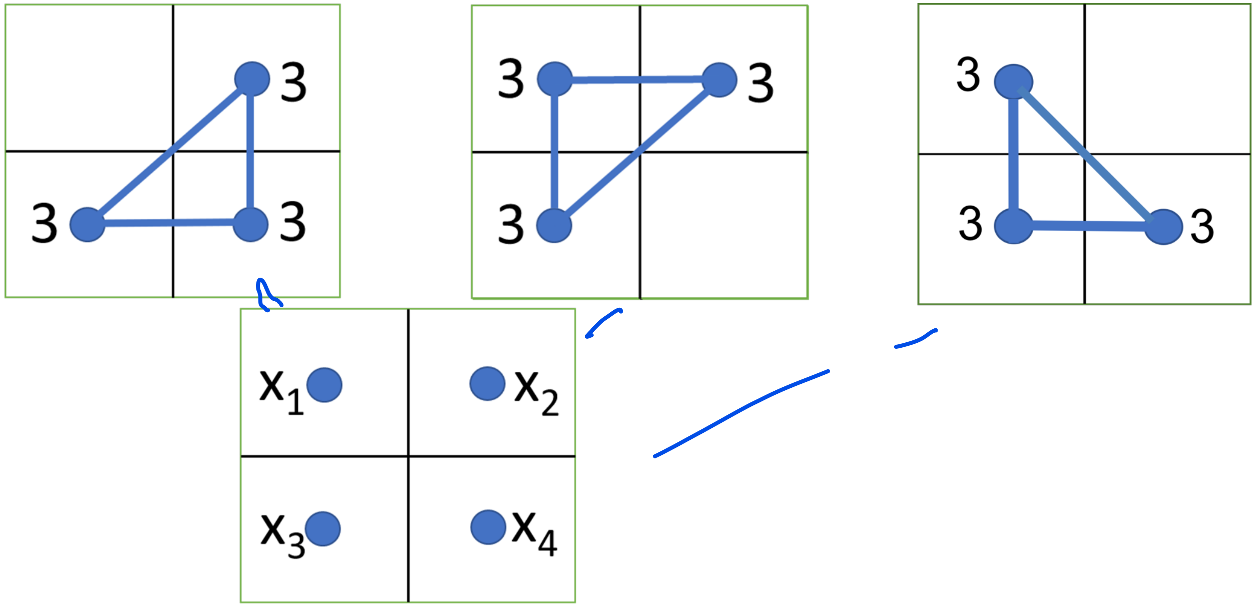
as long as $\mathbf{y}^*$ remains the dual optimal solution for $\mathbf{b}^+$, because

$$z(\mathbf{b}^+) = (\mathbf{b}^+)^T\mathbf{y}^* = z(\mathbf{b}) + \delta \cdot y_k^*.$$

Thus, the optimal dual value is the rate of the net change of the optimal objective value over the net change of an entry of the right-hand-vector resources, i.e.,

$$\nabla z(\mathbf{b}) = \mathbf{y}^*.$$

## Application in the Wassestein Barycenter Problem




Find distribution of $x_i, i = 1, 2, 3, 4$ to minimize

$$\min \quad WD_l(\mathbf{x}) + WD_m(\mathbf{x}) + WD_r(\mathbf{x})$$

$$\text{s.t.} \quad x_1 + x_2 + x_3 + x_4 = 9, \qquad x_i \geq 0,\ i = 1, 2, 3, 4.$$

The objective is a nonlinear function, but its gradient vector $\nabla WD_l(\mathbf{x})$, $\nabla WD_m(\mathbf{x})$ and $\nabla WD_l(\mathbf{x})$ are shadow prices of the three sub-transportation problems –popularly used in Hierarchy Optimization.

## The Dual of the Reinforcement Learning LP

Recall the cost-to-go value of the reinforcement learning LP problem:

primal

$$
\begin{array}{lrcl}
\text{maximize}_{\mathbf{y}} & \sum_{i=1}^{m} y_i & & \\
\text{subject to} & y_1 - \gamma \mathbf{p}_j^T \mathbf{y} & \leq & c_j, \; j \in \mathcal{A}_1 \\
 & & \dots & \\
 & y_i - \gamma \mathbf{p}_j^T \mathbf{y} & \leq & c_j, \; j \in \mathcal{A}_i \\
 & & \dots & \\
 & y_m - \gamma \mathbf{p}_j^T \mathbf{y} & \leq & c_j, \; j \in \mathcal{A}_m.
\end{array}
$$

dual

$$
\begin{array}{lccccc}
\text{minimize}_{\mathbf{x}} & \sum_{j \in \mathcal{A}_1} c_j x_j + & \dots & + \sum_{j \in \mathcal{A}_m} c_j x_j & & \\
\text{subject to} & \sum_{j \in \mathcal{A}_1} (\mathbf{e}_1 - \gamma \mathbf{p}_j) x_j + & \dots & + \sum_{j \in \mathcal{A}_m} (\mathbf{e}_m - \gamma \mathbf{p}_j) x_j & = & \mathbf{e}, \\
 & \dots & x_j & \dots & \geq & 0, \; \forall j,
\end{array}
$$

where $\mathbf{e}_i$ is the unit vector with $1$ at the $i$th position and $0$ everywhere else.

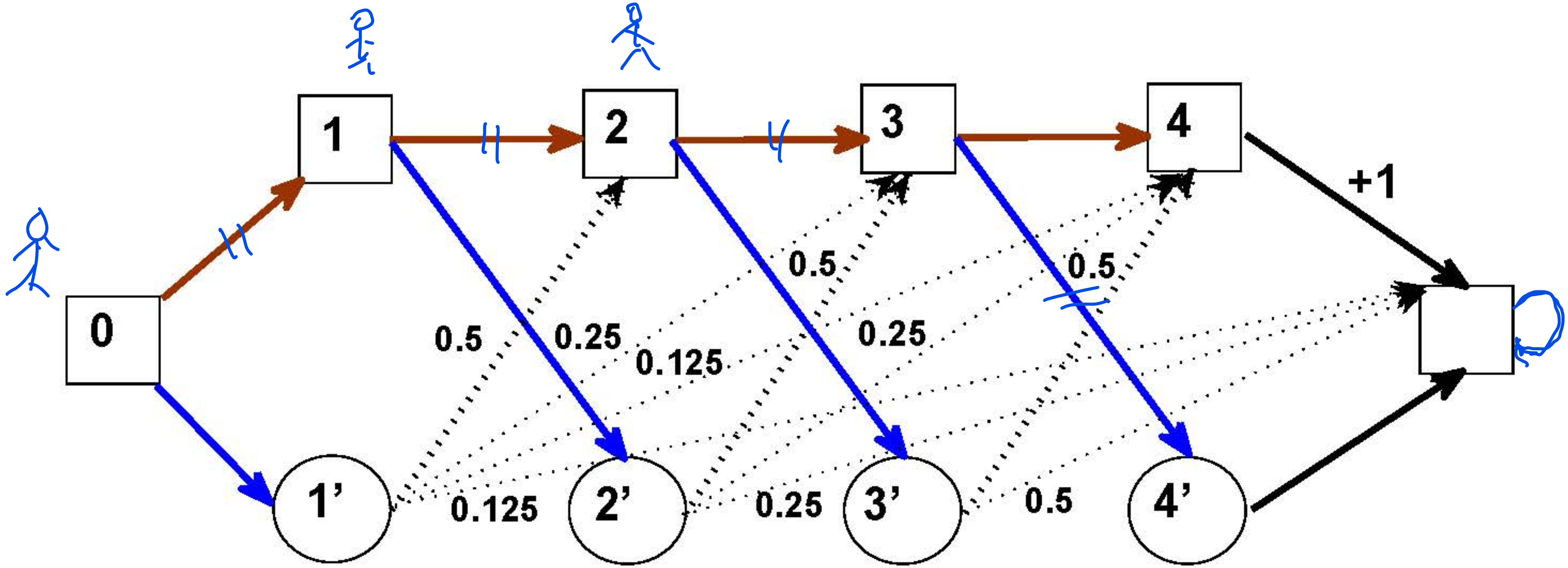
## Interpretation of the Dual of the RL-LP

Variable $x_j$, $j \in \mathcal{A}_i$, is the state-action frequency or called flux, or the expected present value of the number of times that an individual is in state $i$ and takes state-action $j$.

Thus, solving the problem entails choosing a state-action frequencies/fluxes that minimizes the expected present value of total costs for the infinite horizon, where the RHS is $(1;1;1;1;1;1)$:

| x: | $(0_1)$ | $(0_2)$ | $(1_1)$ | $(1_2)$ | $(2_1)$ | $(2_2)$ | $(3_1)$ | $(3_2)$ | $(4_1)$ | $(5_1)$ | **b** |
|---|---|---|---|---|---|---|---|---|---|---|---|
| c: | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $1$ | $0$ | |
| $(0)$ | $1$ | $1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $1$ |
| $(1)$ | $-\gamma$ | $0$ | $1$ | $1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $1$ |
| $(2)$ | $0$ | $-\gamma/2$ | $-\gamma$ | $0$ | $1$ | $1$ | $0$ | $0$ | $0$ | $0$ | $1$ |
| $(3)$ | $0$ | $-\gamma/4$ | $0$ | $-\gamma/2$ | $-\gamma$ | $0$ | $1$ | $1$ | $0$ | $0$ | $1$ |
| $(4)$ | $0$ | $-\gamma/8$ | $0$ | $-\gamma/4$ | $0$ | $-\gamma/2$ | $-\gamma$ | $0$ | $1$ | $0$ | $1$ |
| $(5)$ | $0$ | $-\gamma/8$ | $0$ | $-\gamma/4$ | $0$ | $-\gamma/2$ | $0$ | $-\gamma$ | $-\gamma$ | $1-\gamma$ | $1$ |

where state $5$ is the absorbing state that has a infinite loops to itself.




The optimal dual solution is

$$x^*_{01} = 1,\ x^*_{11} = 1 + \gamma,\ x^*_{21} = 1 + \gamma + \gamma^2,\ x^*_{32} = 1 + \gamma + \gamma^2 + \gamma^3,\ x^*_{41} = 1,$$
$$x^*_{51} = \frac{1+2\gamma+\gamma^2+\gamma^3+\gamma^4}{1-\gamma}.$$

## The Maze Runner Example: Complementarity Condition

The LP optimal Cost-to-Go values are $y_1^* = 0, y_1^* = 0, y_2^* = 0, y_3^* = 0, y_4^* = 1$:

$$
\begin{array}{lrl}
\text{maximize}_{\mathbf{y}} & y_0 + y_1 + y_2 + y_3 + y_4 + y_5 & \\
\text{subject to} & y_0 - \gamma y_1 & \leq 0, \ (x_{01}^* = 1) \\
& y_0 - \gamma(0.5y_2 + 0.25y_3 + 0.125y_4) & \leq 0, \ (x_{02}^* = 0) \\
& y_1 - \gamma y_2 & \leq 0, \ (x_{11}^* = 1 + \gamma) \\
& y_1 - \gamma(0.5y_3 + 0.25y_4) & \leq 0, \ (x_{12}^* = 0) \\
& y_2 - \gamma y_3 & \leq 0, \ (x_{21}^* = 1 + \gamma + \gamma^2) \\
& y_2 - \gamma(0.5y_4) & \leq 0, \ (x_{22}^* = 0) \\
& y_3 - \gamma y_4 & \leq 0, \ (x_{31}^* = 0) \\
& y_3 & \leq 0, \ (x_{32}^* = 1 + \gamma + \gamma^2 + \gamma^3) \\
& y_4 - \gamma y_5 & \leq 1, \ (x_{41}^* = 1) \\
& y_5 - \gamma y_5 & = 0. \ (x_{51}^* = \frac{1+2\gamma+\gamma^2+\gamma^3+\gamma^4}{1-\gamma})
\end{array}
$$

# Dual of Information Markets

$$
\begin{array}{lrl}
\max & \pi^T \mathbf{x} - z & \\
\text{s.t.} & A\mathbf{x} - \mathbf{e}\cdot z & \le \mathbf{0}, \quad \leftarrow p \\
 & \mathbf{x} & \le \mathbf{q}, \quad \leftarrow y \\
 & \mathbf{x} & \ge 0.
\end{array}
$$

$\pi^T \mathbf{x}$: the optimistic amount can be collected.

$z$: the worst-case amount need to pay to the winning bids.

$$
\begin{array}{lrl}
\min & \mathbf{q}^T\mathbf{y} & \\
\text{s.t.} & A^T\mathbf{p} + \mathbf{y} & \ge \pi, \\
 & \mathbf{e}^T\mathbf{p} & = 1, \\
 & (\mathbf{p}, \mathbf{y}) & \ge 0.
\end{array}
$$

$= \sum_{j=1}^{n} q_j y_j$

$y_j \ge \pi_j - a_j^T p$

$\sum p_i = 1$

$p \ge 0, \quad y_j \ge 0 \ \forall j$

$\mathbf{p}$ represents the state prices or probability distributions.

## Dual Interpretation: Regression using Important Data Samples

Note that

$$y_j = \max\{0,\ \pi_j - \mathbf{a}_j^T \mathbf{p}\},\ \forall j.$$

so that

$$\begin{array}{lrl} \min & \sum_j \max\{0,\ \pi_j - \mathbf{a}_j^T \mathbf{p}\} & \\ \text{s.t.} & \mathbf{e}^T \mathbf{p} & = 1, \\ & \mathbf{p} & \geq 0. \end{array}$$

$x_j = q_j, \quad \pi_j - a_j^T p^* > 0$

$0 < x_j < q_j \quad \pi_j - a_j^T p^* = 0$

$x_j = 0, \quad \pi_j - a_j^T p^* < 0$

The $\max\{0,\ \cdot\}$ is called ReLu function in AI.

Dual Interpretation: Find the probability estimations such that low-bids are automatically uncounted/removed.

## Strictly Complementarity Condition in Information Markets

| | |
|---|---|
| $x_j > 0$ | $\mathbf{a}_j^T\mathbf{p} + y_j = \pi_j$ and $y_j \geq 0$ so that $\mathbf{a}_j^T\mathbf{p} \leq \pi_j$ |
| $0 < x_j < q_j$ | $y_j = 0$ so that $\mathbf{a}_j^T\mathbf{p} = \pi_j$ |
| $x_j = q_j$ | $y_j > 0$ so that $\mathbf{a}_j^T\mathbf{p} < \pi_j$ |
| $x_j = 0$ | $\mathbf{a}_j^T\mathbf{p} + y_j > \pi_j$ and $y_j = 0$ so that $\mathbf{a}_j^T\mathbf{p} > \pi_j$ |

The price is Fair:

$$\mathbf{p}^T(A\mathbf{x} - \mathbf{e} \cdot z) = 0 \quad \text{implies} \quad \mathbf{p}^T A\mathbf{x} = \mathbf{p}^T\mathbf{e} \cdot z = z;$$

that is, the worst case cost equals the worth of total shares. Moreover, if a lower bid wins the auction, so does the higher bid on any same type of bids.

## World Cup Information Market Result

Buyer√

| Order: | $\#1$ | $\#2$ | $\#3$ | $\#4$ | $\#5$ | State Price |
|---|---|---|---|---|---|---|
| Argentina | $1$ | $0$ | $1$ | $1$ | $0$ | $0.2$ |
| Brazil | $1$ | $0$ | $0$ | $1$ | $1$ | $0.35$ |
| Italy | $1$ | $0$ | $1$ | $1$ | $0$ | $0.2$ |
| Germany | $0$ | $1$ | $0$ | $1$ | $1$ | $0.25$ |
| France | $0$ | $0$ | $1$ | $0$ | $0$ | $0$ |
| Bidding Price: $\pi$ | $0.75$ | $0.35$ | $0.4$ | $0.95$ | $0.75$ | |
| Quantity limit: $\mathbf{q}$ | $10$ | $5$ | $10$ | $10$ | $5$ | |
| Order fill: $\mathbf{x}^*$ | $5$ | $5$ | $5$ | $0$ | $5$ | |

Question: How to make the dual prices unique and the market online?

$w_t \ln(u_t)$

## General Auction-Market

t th bid

Consider problem:

$$\begin{array}{lll} \text{maximize}_{\mathbf{x}} & \sum_{t=1}^{n} \pi_t x_t \ (\text{or } u_t(x_t)) & \\ \text{subject to} & \sum_{t=1}^{n} a_{it} x_t \leq b_i, \quad p & \forall i = 1, ..., m \\ & 0 \leq x_t \leq 1, \quad x_t \in \{0,1\} & \forall t = 1, ..., n \end{array}$$

Each bid/activity $t$ requests a bundle of $m$ resources, and the payment is $\pi_t$.

$n >> m$

| | order 1($t = 1$) | order 2($t = 2$) | ..... | Inventory($\mathbf{b}$) |
|---|---|---|---|---|
| Price($\pi_t$) | \$100 | \$30 | ... | |
| Decision | $x_1$ | $x_2$ | ... | |
| Pants | 1 | 0 | ... | 100 |
| Shoes | - 1 | 0 | ... | 50 |
| T-shirts | 0 | 1 | ... | 500 |
| Jacket | 0 | 0 | ... | 200 |
| Socks | 1 | 1 | ... | 1000 |

via

P

m

## Dual of the Auction-Market Problem

$$\begin{array}{ll} \text{minimize}_{\mathbf{x}} & \mathbf{b}^T\mathbf{p} + \sum_{\cancel{j}\, t=1}^{n} z_j \\ \text{subject to} & \cancel{\mathbf{p}^T\mathbf{a}_t - \pi_t + z_t \geq 0 \quad \forall \cancel{j}\, t = 1, ..., n} \\ & (\mathbf{p}, \cancel{\mathbf{z}}) \geq \mathbf{0} \end{array}$$

$\sum_{t=1}^{n} \max\{0, \pi_t - a_t^T p\}$

$z_t \geq \pi_t - a_t^T p$

$z_t \geq 0$

Strict Complementarity/Optimality Conditions:

n-d:

$$x_t = \begin{cases} 0 & \text{if } \pi_t < \mathbf{p}^T\mathbf{a}_t \\ 1 & \text{if } \pi_t > \mathbf{p}^T\mathbf{a}_t \\ (0\ 1) & \text{if } \pi_t \underline{=} \mathbf{p}^T\mathbf{a}_t \end{cases}$$

$p \rightarrow m$

$\mathbf{p}$ are itemized prices of Goods!

## Sensor Network Localization

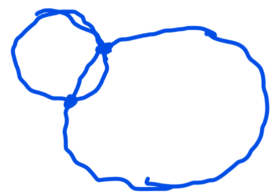
Recall the system of nonlinear equations for $\mathbf{x}_i \in R^d$:

$$\|\mathbf{x}_i - \mathbf{x}_j\|_2 = d_{ij},\ \forall\,(i,j) \in N_x,\ i < j,$$
$$\|\mathbf{a}_k - \mathbf{x}_j\|_2 = \bar{d}_{kj},\ \forall\,(k,j) \in N_a,$$

where $\mathbf{a}_k$ are possible points whose locations are known, often called anchors.

One can equivalently represent it as

QCQP
$$\begin{cases} \|\mathbf{x}_i - \mathbf{x}_j\|_2^2 = d_{ij}^2,\ \forall\,(i,j) \in N_x,\ i < j, \\ \|\mathbf{a}_k - \mathbf{x}_j\|_2^2 = \bar{d}_{kj}^2,\ \forall\,(k,j) \in N_a, \end{cases}$$



which becomes a system of multi-variable-quadratic equations.

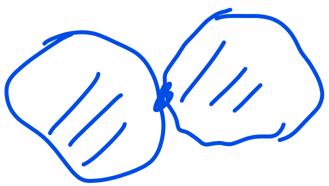
## SOCP Relaxation for SNL

System of SOCP Feasibility for $\mathbf{x}_i \in R^2$:

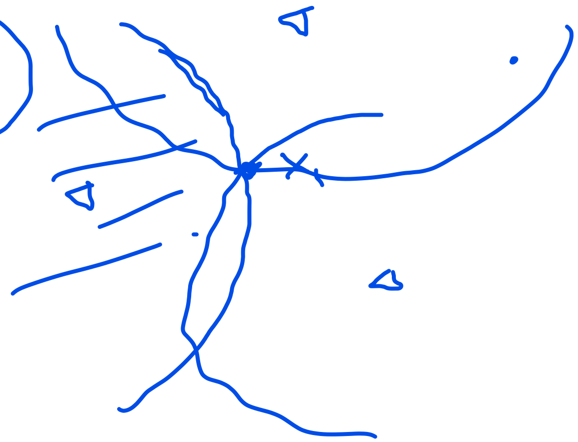
$$\left\{\begin{array}{l} \|\mathbf{x}_i - \mathbf{x}_j\| \le d_{ij}, \; \forall\, (i,j) \in N_x, \; i < j, \\ \|\mathbf{a}_k - \mathbf{x}_j\| \le d_{kj}, \; \forall\, (k,j) \in N_a, \end{array}\right.$$

where $\mathbf{a}_k$ are points whose locations are known.

Consider the case where a single unknown point $\mathbf{x}_1$ is connected to three anchors $\mathbf{a}_k, \; k = 1, 2, 3$ on $R^2$:



$$\|\mathbf{a}_k - \mathbf{x}\| \le d_k, \; k = 1, 2, 3$$



## The Standard SOCP Relaxation and Dual

$$
\begin{array}{ll}
\text{minimize} & 0 \\
& \delta_k = d_k, \ (\lambda_k), \ k = 1, 2, 3 \\
& \mathbf{y}_k + \mathbf{x} = \mathbf{a}_k, \ (\mathbf{z}_k), \ k = 1, 2, 3 \\
& (\delta_k; \mathbf{y}_k) \in SOCP, \ k = 1, 2, 3
\end{array}
$$

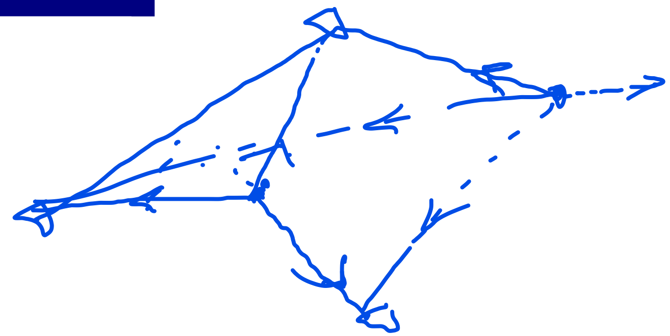
The Dual

$$
\begin{array}{ll}
\text{maximize} & \sum_k (d_k \lambda_k + \mathbf{a}_k^T \mathbf{z}_k) \\
& \sum_k \mathbf{z}_k = \mathbf{0}, \\
& (-\lambda_k; -\mathbf{z}_k) \in SOCP, \ k = 1, 2, 3
\end{array}
$$

Suppose the true sensor location is $\mathbf{b}$, the dual can be written as

$$
\begin{array}{ll}
\text{minimize} & \sum_k (-d_k \lambda_k + (\mathbf{a}_k - \mathbf{b})^T \mathbf{z}_k) \\
& \sum_k \mathbf{z}_k = \mathbf{0}, \\
& (\lambda_k; \mathbf{z}_k) \in SOCP, \ k = 1, 2, 3
\end{array}
$$

## Optimality Condition of the SOCP Relaxation



The conditions would be

$$\mathbf{z}_k = (\lambda_k/d_k)(\mathbf{a}_k - \mathbf{b})$$

and

$$\sum_k (\lambda_k/d_k)(\mathbf{a}_k - \mathbf{b}) = \mathbf{0}$$

Thus, $\lambda_k$ represents a positive force in direction $\mathbf{a}_k - \mathbf{b}$, and the total forces should be balanced along the three directions.

If $\mathbf{b}$ is in the convex-hull, this can be achieved so that the optimal solution of the SOCP relaxation is $\mathbf{x}^* = \mathbf{b}$.

What happen if NOT?

## SDP Relaxation for SNL

Find a symmetric matrix $Z \in \mathbf{R}^{(2+n)\times(2+n)}$ such that

$$\begin{aligned}
Z_{1:2,1:2} &= I \\
(\mathbf{0}; \mathbf{e}_i - \mathbf{e}_j)(\mathbf{0}; \mathbf{e}_i - \mathbf{e}_j)^T \bullet Z &= d_{ij}^2, \ \forall\, i, j \in N_x,\ i < j, \\
(\mathbf{a}_k; -\mathbf{e}_j)(\mathbf{a}_k; -\mathbf{e}_j)^T \bullet Z &= d_{kj}^2, \ \forall\, k, j \in N_a, \\
Z &\succeq \mathbf{0}.
\end{aligned}$$

This is semidefinite programming feasibility system (with a null objective).

When this relaxation is exact?

One case is that the single unknown point $\mathbf{x}_1$ is connected to three anchors $\mathbf{a}_k,\ k = 1, 2, 3$.

In general, if the rank of a feasible $Z$ is $2$, then it solves the original graph relaxation problem.

## Duality Theorem for SNL

**Theorem 1** *Let $\bar{Z}$ be a feasible solution for SDP and $\bar{U}$ be an optimal slack matrix of the dual. Then,*

1. *complementarity condition holds: $\bar{Z} \bullet \bar{U} = 0$ or $\bar{Z}\bar{U} = \mathbf{0}$;*
2. *$Rank(\bar{Z}) + Rank(\bar{U}) \leq 2 + n$;*
3. *$Rank(\bar{Z}) \geq 2$ and $Rank(\bar{U}) \leq n$.*

$n+2$

$$\begin{pmatrix} \boxed{I} & X \\ X & Y \end{pmatrix} = Z$$

An immediate result from the theorem is the following:

**Corollary 1** *If an optimal dual slack matrix has rank $n$, then every solution of the SDP has rank $2$, that is, the SDP relaxation solves the original problem exactly.*

## Theoretical Analyses on SNL-SDP Relaxation

A sensor network is $2$-universally-localizable (UL) if there is a unique localization in $\mathbf{R}^2$ and there is no $x_j \in \mathbf{R}^h,\ j = 1, ..., n$, where $h > 2$, such that

$$\|x_i - x_j\|^2 = d_{ij}^2,\ \forall\, i, j \in N_x,\ i < j,$$
$$\|(a_k; \mathbf{0}) - x_j\|^2 = \hat{d}_{kj}^2,\ \forall\, k, j \in N_a.$$

The latter says that the problem cannot be localized in a higher dimension space where anchor points are simply augmented to $(a_k; \mathbf{0}) \in \mathbf{R}^h$, $k = 1, ..., m$.

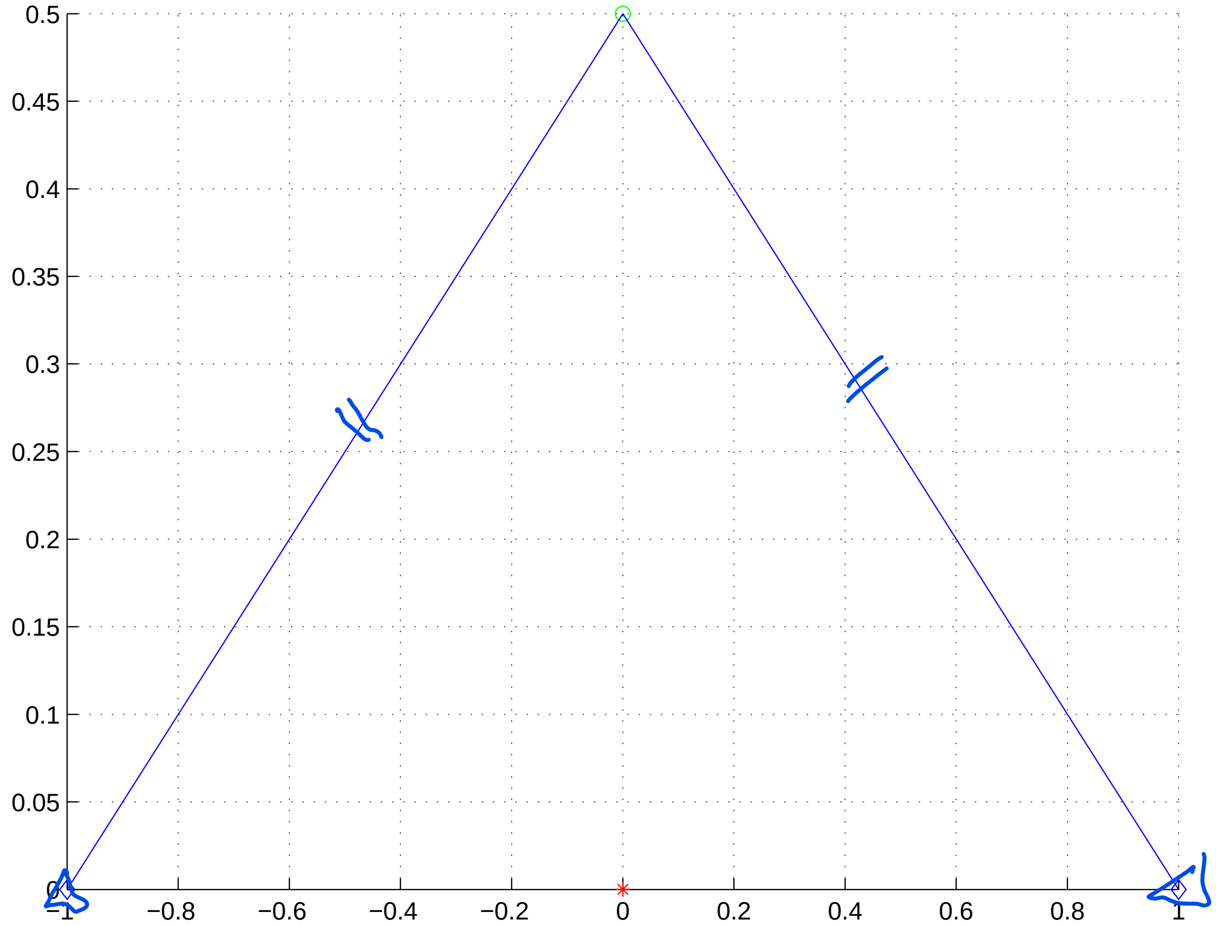



Figure 1: One sensor-Two anchors: Not Localizable

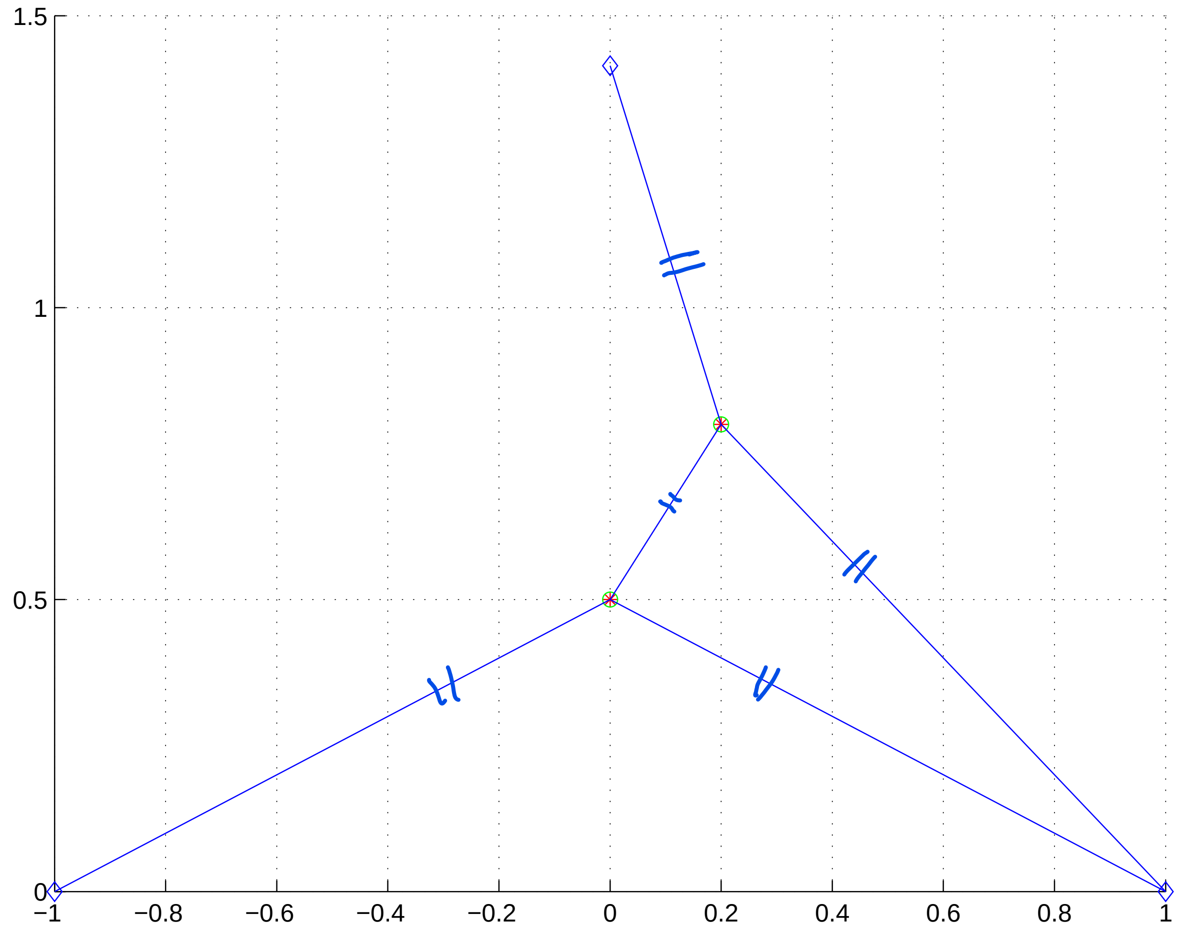



Figure 2: Two sensor-Three anchors: Strongly Localizable

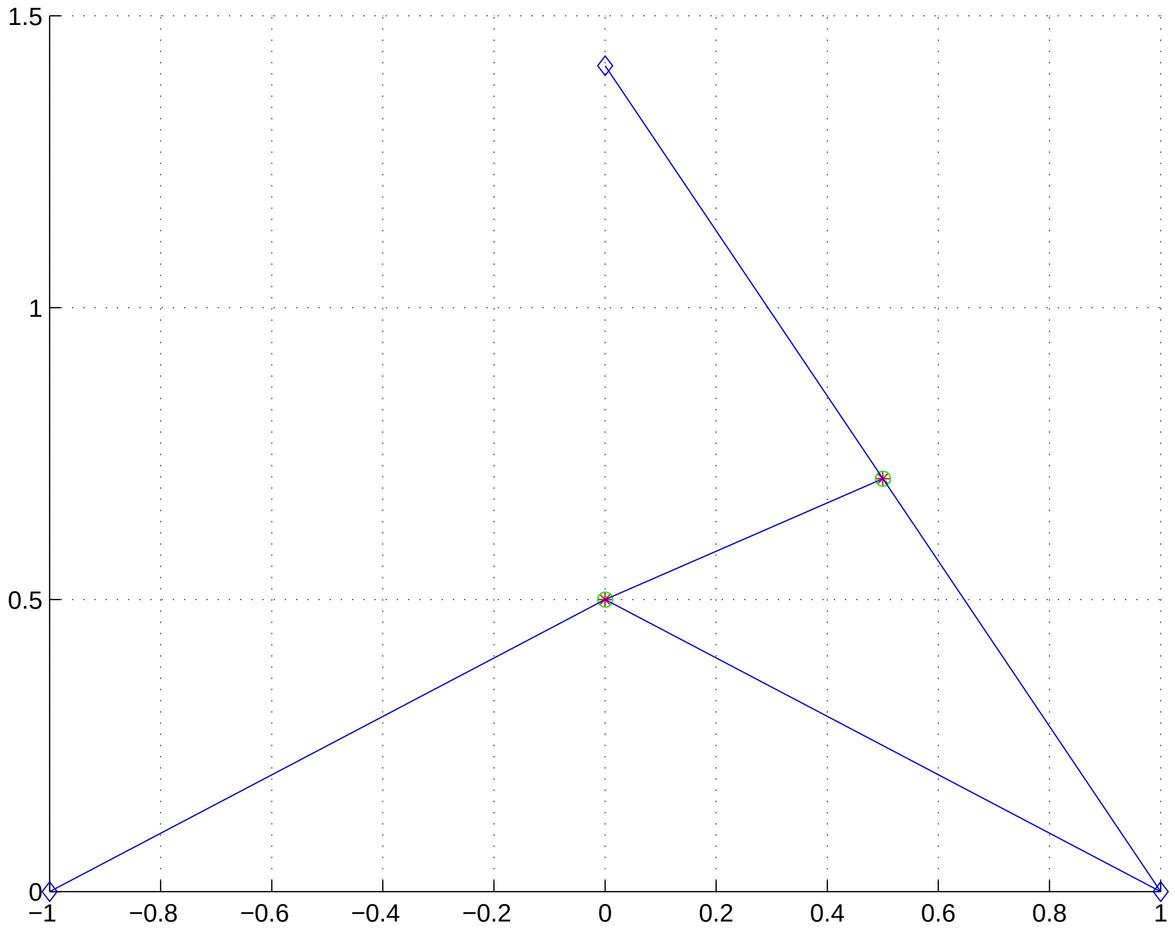



Figure 3: Two sensor-Three anchors: Localizable but not Strongly

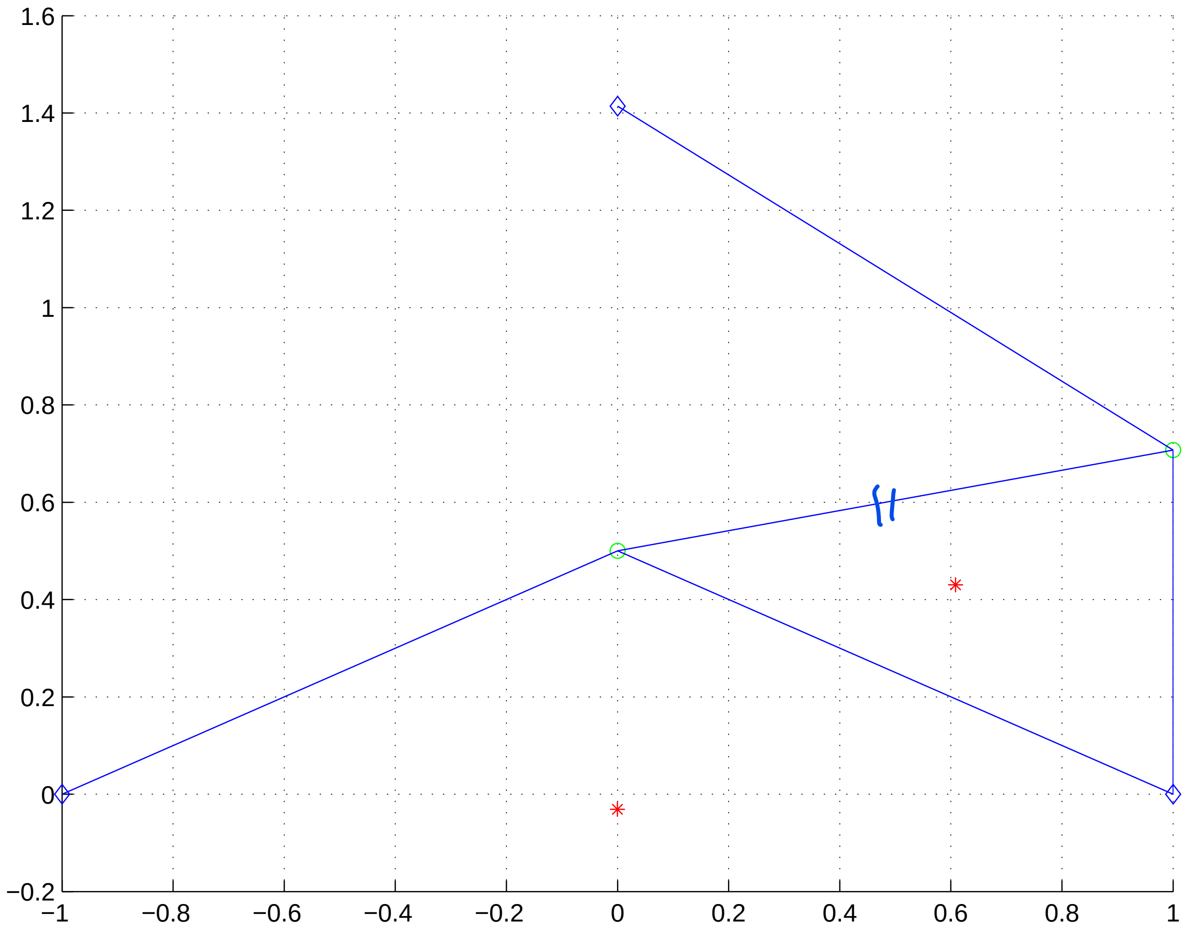



Figure 4: Two sensor-Three anchors: Not Localizable

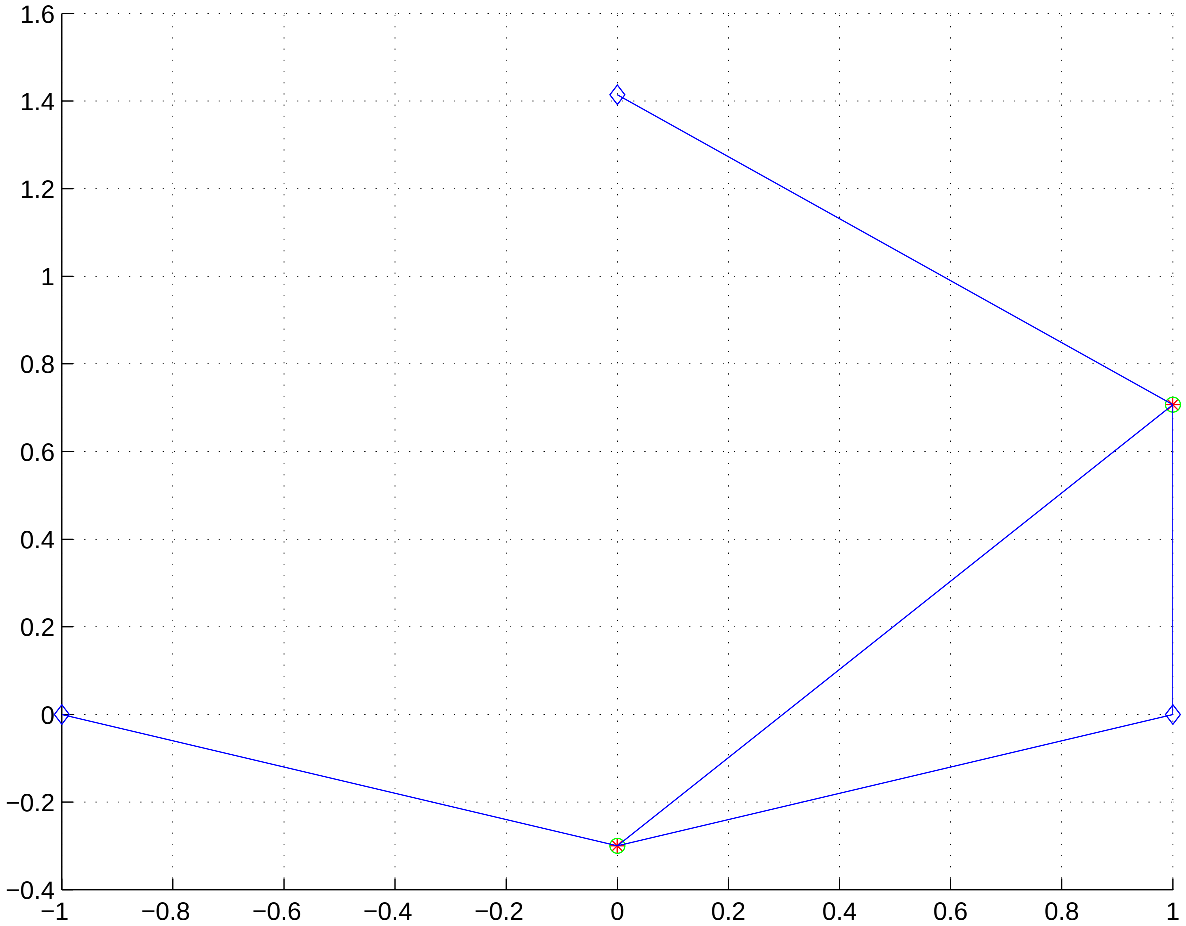



Figure 5: Two sensor-Three anchors: Strongly Localizable

## Universally-Localizable Problems (ULP)

**Theorem 2** *The following SNL problems are Universally-Localizable:*

- *If every edge length is specified, then the sensor network is $2$-universally-localizable (Schoenberg 1942).*
- *There is a sensor network (trilateral graph), with $O(n)$ edge lengths specified, that is $2$-universally-localizable (So 2007).*
- *If one sensor with its edge lengths to at least three anchors (in general positions) specified, then it is $2$-universally-localizable (So and Y 2005).*

## ULPs Can be Localized as Convex Optimization

**Theorem 3** *(So and Y 2005) The following statements are equivalent:*

1. *The sensor network is $2$-universally-localizable;*
2. *The max-rank solution of the SDP relaxation has rank $2$;*
3. *The solution matrix has $Y = X^T X$ or Tr$(Y - X^T X) = 0$ .*

$\mathrm{rank}(Z) = 2$

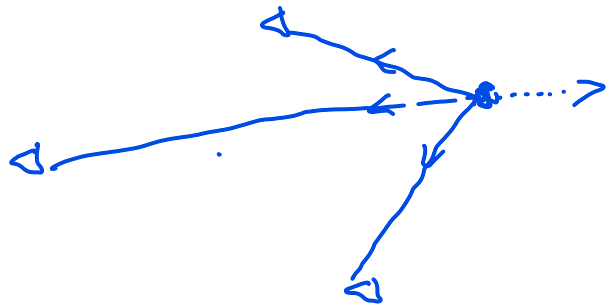
$Y \succeq$

When an optimal dual (stress) slack matrix has rank $n$, then the problem is $2$-strongly-localizable-problem (SLP). This is a sub-class of ULP.

Example: if one sensor with its edge lengths to three anchors (in general positions) are specified, then it is $2$-strongly-localizable.

SOCP

## One Sensor and three Anchors



(≤)

Find $x_1 \in \mathbf{R}^2$ such that

$$\|\mathbf{a}_k - \mathbf{x}_1\|^2 = \hat{d}_{kj}^2, \text{ for } k = 1, 2, 3,$$

Let $\bar{\mathbf{x}}_1$ be the true position of the sensor.

## SDP Relaxation Standard Form

$$
\begin{aligned}
&(1;0;0)(1;0;0)^T \bullet Z = 1,\\
&(0;1;0)(0;1;0)^T \bullet Z = 1,\\
&(1;1;0)(1;1;0)^T \bullet Z = 2,\\
&(\mathbf{a}_k;-1)(\mathbf{a}_k;-1)^T \bullet Z = \hat{d}_{k1}^2, \text{ for } k = 1,2,3,\\
&Z \succeq \mathbf{0}.
\end{aligned}
$$

$$
\bar{Z} = \begin{pmatrix} I & \bar{\mathbf{x}}_1 \\ \bar{\mathbf{x}}_1^T & \bar{x}_1^T \bar{x}_1 \end{pmatrix} = (I,\ \bar{\mathbf{x}}_1)^T (I,\ \bar{\mathbf{x}}_1)
$$

is a feasible rank-2 solution for the relaxation.

## Dual Slack Matrices

$$\begin{pmatrix} \begin{pmatrix} w_1 + w_3 & w_3 \\ w_3 & w_2 + w_3 \end{pmatrix} + \sum_{k=1}^{3} \hat{w}_{k1} \mathbf{a}_k \mathbf{a}_k^T & -\sum_{k=1}^{3} \hat{w}_{k1} a_k \\ -(\sum_{k=1}^{3} \hat{w}_{k1} a_k)^T & \hat{w}_{11} + \hat{w}_{21} + \hat{w}_{31} \end{pmatrix} \succeq \mathbf{0}.$$

Does an optimal slack matrix $U$ have rank $1$ with

$$w_1 + w_2 + 2w_3 + \sum_{k=1}^{3} \hat{w}_{k1} \hat{d}_{k1}^2 = 0?$$

## Optimal Dual Slack Matrix

If we choose $w_\bullet$'s such that

$$\bar{U} = (-\bar{x}_1; 1)(-\bar{\mathbf{x}}_1; 1)^T,$$

then, $\bar{U} \succeq \mathbf{0}$ and $\bar{U} \bullet \bar{X} = 0$ so that $\bar{U}$ is an optimal slack matrix for the dual and its rank is $1$.

## How to Select $w$'s

We only need to consider choosing $\hat{w}$'s:

$$\sum_{k=1}^{3} \hat{w}_{k1}\mathbf{a}_k = \bar{\mathbf{x}}_1 \qquad\qquad \sum_{k=1}^{3} \hat{w}_{k1}(\mathbf{a}_k - \bar{\mathbf{x}}_1) = \mathbf{0}$$

or

$$\hat{w}_{11} + \hat{w}_{21} + \hat{w}_{31} = 1. \qquad\qquad \hat{w}_{11} + \hat{w}_{21} + \hat{w}_{31} = 1.$$

This system always has a solution if $\mathbf{a}_k$ is not co-linear.

Then, select the rest

$$\begin{pmatrix} w_1 + w_3 & w_3 \\ w_3 & w_2 + w_3 \end{pmatrix} = \bar{\mathbf{x}}_1\bar{\mathbf{x}}_1^T - \sum_{k=1}^{3} \hat{w}_{k1}\mathbf{a}_k\mathbf{a}_k^T$$

## Other Conditions?

Even if $\mathbf{a}_k$ is co-linear, the system

$$\sum_{k=1}^{3} \hat{w}_{k1}(\mathbf{a}_k - \bar{\mathbf{x}}_1) = \mathbf{0}$$

$$\hat{w}_{11} + \hat{w}_{21} + \hat{w}_{31} = 1$$

may still have a solution $w_\bullet$?

Physical interpretation: $\hat{w}_{kj}$ is a stress/force on the edge and all stresses are balanced or at an equilibrium state. The objective represents the potential of the system.

## Localize All Localizable Points

**Theorem 4** *(So and Y 2005) If a problem (graph) contains a subproblem (subgraph) that is universally-localizable, then the submatrix solution corresponding to the subproblem in the SDP solution has rank $2$. That is, the SDP relaxation computes a solution that localize all possibly localizable unknown sensor points.*

The proof is similar to the proof of Theorem 3 by removing the notes that is not localizable.

Implication: Diagonals of “co-variance” matrix

$$\bar{Y} - \bar{X}^T \bar{X},$$

$\bar{Y}_{jj} - \|\bar{x}_j\|^2$, can be used as a measure to see whether $j$th sensor’s estimated position is reliable or not.

# Uncertainty Analysis and Confidence Measure

Alternatively, each $x_j$'s can be viewed as uncertain points from the incomplete/uncertain distance measures. Then the solution to the SDP problem provides the first and second moment estimation (Bertsimas and Y 1998).

Generally, $\bar{x}_j$ is a point estimate of $x_j$ and $\bar{Y}_{ij}$ is a point estimate $x_i^T x_j$.

Consequently,

$$\bar{Y}_{jj} - \|\bar{x}_j\|^2,$$

which is the individual variance estimation of sensor $j$, gives an interval estimation for its true position (Biswas and Y 2004).