Conic Duality Theorems and Applications

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Chapters 3.1-3.2 and 6.1-6.4

Primal and Dual of Conic LP

Recall the pair of

$$
\begin{array}{lll}
(CLP) & \text{minimize} & \mathbf{c} \bullet \mathbf{x} \\
& \text{subject to} & \mathbf{a}_i \bullet \mathbf{x} = b_i, i = 1, 2, ..., m, \ (\mathcal{A}\mathbf{x} = \mathbf{b}), \ \mathbf{x} \in K;
\end{array}
$$

and it dual problem

$$
\begin{array}{lll}
(CLD) & \text{maximize} & \mathbf{b}^T \mathbf{y} \\
& \text{subject to} & \sum_i^m y_i \mathbf{a}_i + \mathbf{s} = \mathbf{c}, \ (\mathcal{A}^T \mathbf{y} + \mathbf{s} = \mathbf{c}), \ \mathbf{s} \in K^*,
\end{array}
$$

where $\mathbf{y} \in \mathcal{R}^m$, $\mathbf{s}$ is called the dual slack vector/matrix, and K^* is the dual cone of K.

Theorem 1 *(Weak duality theorem)* $\mathbf{c} \bullet \mathbf{x} - \mathbf{b}^T \mathbf{y} = \mathbf{x} \bullet \mathbf{s} \geq 0$ *for any feasible* $\mathbf{x}$ *of (CLP) and* $(\mathbf{y}, \mathbf{s})$ *of (CLD).*

Here, operator $\mathcal{A}\mathbf{x}$ and Adjoint-Operator $\mathcal{A}^T\mathbf{y}$ minimic matrix-vector production $A\mathbf{x}$ and its transpose operation $A^T\mathbf{y}$, where

$$
\mathcal{A} = (\mathbf{a}_1; \mathbf{a}_2; ...; \mathbf{a}_m), \quad \mathcal{A}\mathbf{x} = (\mathbf{a}_1 \bullet \mathbf{x}; ...; \mathbf{a}_m \bullet \mathbf{x}), \quad \text{and} \quad \mathcal{A}^T \mathbf{y} = \sum_i y_i \mathbf{a}_i^T.
$$

CLP Duality Theorems

The weak duality theorem shows that a feasible solution to either problem yields a bound on the value of the other problem. We call $\mathbf{c} \bullet \mathbf{x} - \mathbf{b}^T\mathbf{y}$ the duality gap.

Corollary 1 *Let $\mathbf{x}^* \in \mathcal{F}_p$ and $(\mathbf{y}^*, \mathbf{s}^*) \in \mathcal{F}_d$. Then, $\mathbf{c} \bullet \mathbf{x}^* = \mathbf{b}^T\mathbf{y}^*$ implies that $\mathbf{x}^*$ is optimal for (CLP) and $(\mathbf{y}^*, \mathbf{s}^*)$ is optimal for (CLD).*

Is the reverse also true? That is, given $\mathbf{x}^*$ optimal for (CLP), then there is $(\mathbf{y}^*, \mathbf{s}^*)$ feasible for (CLD) and $\mathbf{c} \bullet \mathbf{x}^* = \mathbf{b}^T\mathbf{y}^*$?

This is called the Strong Duality Theorem.

“True” when $K = \mathcal{R}^n_+$, that is, the polyhedral cone case, but not true in general.

Proof of Strong Duality Theorem for LP

Let $\mathbf{x}^*$ be an minimizer of (LP). Then the following system

$$A\mathbf{x}' - \mathbf{b}\tau = \mathbf{0}, \quad (\mathbf{x}';\tau) \geq \mathbf{0}, \quad \mathbf{c}^T\mathbf{x}' - (\mathbf{c}^T\mathbf{x}^*)\tau = -1 < 0$$

must have no feasible solution $(\mathbf{x}';\tau)$. This is because otherwise, if $\tau > 0$, $\mathbf{x}'/\tau$ is feasible for (LP) and $\mathbf{c}^T\mathbf{x}'/\tau < \mathbf{c}^T\mathbf{x}^*$, which is a contradiction; and if $\tau = 0$, $\mathbf{x}^* + \mathbf{x}'$ is feasible for (LP) and $\mathbf{c}^T(\mathbf{x}^* + \mathbf{x}') = \mathbf{c}^T\mathbf{x}^* - 1 < \mathbf{c}^T\mathbf{x}^*$, which is also a contradiction. Thus, from the LP alternative system pair II, there is $\mathbf{y}^*$ feasible for

$$\mathbf{c} - A^T\mathbf{y}^* \geq \mathbf{0}, \quad -\mathbf{c}^T\mathbf{x}^* + \mathbf{b}^T\mathbf{y}^* \geq 0.$$

Then, $\mathbf{y}^*$ is feasible for (LD) from the first inequality; and from the weak duality theorem and the second inequality $\mathbf{c}^T\mathbf{x}^* - \mathbf{b}^T\mathbf{y}^* = 0$.

LP and LD Cases

Theorem 2 *The following statements hold for every pair of (LP) and (LD) :*

i) *If (LP) and (LD) are both feasible, then both problems have optimal solutions and the optimal objective values of the objective functions are equal, that is, optimal solutions for both (LP) and (LD) exist and there is no duality gap.*

ii) *If (LP) or (LD) is feasible and bounded, then the other is feasible and bounded.*

iii) *If (LP) or (LD) is feasible and unbounded, then the other has no feasible solution.*

iv) *If (LP) or (LD) is infeasible, then the other is either unbounded or has no feasible solution.*

A case that neither (LP) nor (LD) is feasible: $\mathbf{c} = (-1;\ 0), \quad A = (0,\ -1), \quad b = 1.$

The proofs follow the Farkas lemma and the Weak Duality Theorem.

The LP Primal and Dual Relation

Dual \ Primal	F-B	F-UB	IF
F-B	☺		
F-UB			☹
IF		☹	☹

$$
\begin{array}{llllll}
\min & -x_1 & - & x_2 & & \\
\text{s.t.} & x_1 & - & x_2 & = & 1 \\
 & -x_1 & + & x_2 & = & 1 \\
 & x_1, & & x_2 & \geq & 0
\end{array}
\qquad
\begin{array}{llllll}
\max & y_1 & + & y_2 & & \\
\text{s.t.} & y_1 & - & y_2 & \leq & -1 \\
 & -y_1 & + & y_2 & \leq & -1
\end{array}
$$

Figure 1: Both primal and dual are infeasible

Farkas Lemma and Duality

The Farkas lemma concerns the system the system $\{\mathbf{x} : \ A\mathbf{x} = \mathbf{b}, \ \mathbf{x} \geq \mathbf{0}\}$ and its alternative $\{\mathbf{y} : \ -A^T\mathbf{y} \geq \mathbf{0}, \mathbf{b}^T\mathbf{y} > 0\}$ for given data $(A, \mathbf{b})$. This pair can be represented as a primal-dual LP pair

$$
\begin{array}{ll}
\text{min} & \mathbf{0}^T\mathbf{x} \\
\text{s. t.} & A\mathbf{x} = \mathbf{b}, \\
 & \mathbf{x} \geq \mathbf{0};
\end{array}
\qquad
\begin{array}{ll}
\text{max} & \mathbf{b}^T\mathbf{y} \\
\text{s.t.} & A^T\mathbf{y} \leq \mathbf{0}.
\end{array}
$$

If the primal is infeasible, then the dual must be feasible and unbounded since it is always feasible.

Geometric Interpretation: Let $A = [\mathbf{a}_1, \ \mathbf{a}_2, ..., \mathbf{a}_n]$, then if $\mathbf{b} \notin \{A\mathbf{x} : \ \mathbf{x} \geq \mathbf{0}\}$, then there must be a vector $\mathbf{y}$ where the angle between $\mathbf{y}$ and $\mathbf{b}$ is strictly acute, and the angle with $\mathbf{a}_j$ is either right or obtuse for all i.

Optimality Conditions for LP

$$\left\{ (\mathbf{x}, \mathbf{y}, \mathbf{s}) \in (\mathcal{R}^n_+, \mathcal{R}^m, \mathcal{R}^n_+) : \begin{array}{rcl} \mathbf{c}^T\mathbf{x} - \mathbf{b}^T\mathbf{y} & = & 0 \\ A\mathbf{x} & = & \mathbf{b} \\ -A^T\mathbf{y} - \mathbf{s} & = & -\mathbf{c} \end{array} \right\},$$

which is a system of linear inequalities and equations. Now it is easy to verify whether or not a pair $(\mathbf{x}, \mathbf{y}, \mathbf{s})$ is optimal.

Complementarity Condition

For feasible $\mathbf{x}$ and $(\mathbf{y}, \mathbf{s})$, $\mathbf{x}^T\mathbf{s} = \mathbf{x}^T(\mathbf{c} - A^T\mathbf{y}) = \mathbf{c}^T\mathbf{x} - \mathbf{b}^T\mathbf{y}$ is called the complementarity gap.

If $\mathbf{x}^T\mathbf{s} = 0$, then we say $\mathbf{x}$ and $\mathbf{s}$ are complementary to each other.

Since both $\mathbf{x}$ and $\mathbf{s}$ are nonnegative, $\mathbf{x}^T\mathbf{s} = 0$ implies that $\mathbf{x}_. * \mathbf{s} = \mathbf{0}$ or $x_j s_j = 0$ for all $j = 1, \ldots, n$.

$$\begin{aligned} \mathbf{x}_. * \mathbf{s} &= \mathbf{0} \\ A\mathbf{x} &= \mathbf{b} \\ -A^T\mathbf{y} - \mathbf{s} &= -\mathbf{c}. \end{aligned}$$

This system has total $2n + m$ unknowns and $2n + m$ equations including n nonlinear equations.

Interpretation of $s_j = 0$: the jth inequality constraint of the dual is "binding" or "active".

The Maze Runner Example: Complementarity Condition

The LP optimal Cost-to-Go values are $y_1^* = 0, y_1^* = 0, y_2^* = 0, y_3^* = 0, y_4^* = 1$:

$$
\begin{array}{lrl}
\text{maximize}_{\mathbf{y}} & y_0 + y_1 + y_2 + y_3 + y_4 + y_5 & \\
\text{subject to} & y_0 - \gamma y_1 & \leq 0, \ (x_1^* = 1) \\
& y_0 - \gamma(0.5y_2 + 0.25y_3 + 0.125y_4) & \leq 0, \ (x_2^* = 0) \\
& y_1 - \gamma y_2 & \leq 0, \ (x_3^* = 1 + \gamma) \\
& y_1 - \gamma(0.5y_3 + 0.25y_4) & \leq 0, \ (x_4^* = 0) \\
& y_2 - \gamma y_3 & \leq 0, \ (x_5^* = 1 + \gamma + \gamma^2) \\
& y_2 - \gamma(0.5y_4) & \leq 0, \ (x_6^* = 0) \\
& y_3 - \gamma y_4 & \leq 0, \ (x_7^* = 0) \\
& y_3 & \leq 0, \ (x_8^* = 1 + \gamma + \gamma^2 + \gamma^3) \\
& y_4 - \gamma y_5 & \leq 1, \ (x_9^* = 1) \\
& y_5 - \gamma y_5 & = 0. \ (x_1^*0 = \frac{1+2\gamma+\gamma^2+\gamma^3+\gamma^4}{1-\gamma})
\end{array}
$$

General CLP: an SDP Example with a Duality Gap

The strong duality theorem may not hold for general convex cones:

$$\mathbf{c} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \mathbf{a}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \mathbf{a}_2 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

and

$$\mathbf{b} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$

When Strong Duality Theorems Holds for CLP

Theorem 3 *The following statements hold for every pair of (CLP) and (CLD):*

i) *If (CLP) and (CLD) both are feasible, and furthermore one of them have an interior, then there is no duality gap between (CLP) and (CLD). However, one of the optimal solution may not be attainable.*

ii) *If (CLP) and (CLD) both are feasible and have interior, then, then both have attainable optimal solutions with no duality gap.*

iii) *If (CLP) or (CLD) is feasible and unbounded, then the other has no feasible solution.*

iv) *If (CLP) or (CLD) is infeasible, and furthermore the other is feasible and has an interior, then the other is unbounded.*

In case i), one of the optimal solution may not be attainable although no gap.

SDP Example with Zero-Duality Gap but not Attainable

$$C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \; A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad b_1 = 2.$$

The primal has an interior, but the dual does not.

Proof of CLP Strong Duality Theorem under Nonempty Interior Cond.

i) Let $\mathcal{F}_p$ be feasible and have an interior, and let z^* be its infimum. Then we consider the alternative system pair

$$\mathcal{A}\mathbf{x} - \mathbf{b}\tau = \mathbf{0},\ \mathbf{c} \bullet \mathbf{x} - z^*\tau < 0,\ (\mathbf{x}, \tau) \in K \times R_+,$$

and

$$\mathcal{A}^T\mathbf{y} + \mathbf{s} = \mathbf{c},\ -\mathbf{b}^T\mathbf{y} + \kappa = -z^*,\ (\mathbf{s}, \kappa) \in K^* \times R_+.$$

But the former is infeasible, so that we have a solution for the latter. From the Weak Duality theorem, we must have $\kappa = 0$, that is, we have a solution $(\mathbf{y}, \mathbf{s})$ such that

$$\mathcal{A}^T\mathbf{y} + \mathbf{s} = \mathbf{c},\ \mathbf{b}^T\mathbf{y} = z^*,\ \mathbf{s} \in K^*.$$

ii) We only need to prove that there exist a solution $\mathbf{x} \in \mathcal{F}_p$ such that $\mathbf{c} \bullet \mathbf{x} = z^*$, that is, the infimum of (CLP) is attainable. But this is just the other side of the proof given that $\mathcal{F}_d$ is feasible and has an interior, and z^* is also the supremum of (CLD).

iii) The proof by contradiction follows the Weak Duality Theorem.

iv) Suppose $\mathcal{F}_d$ is empty and $\mathcal{F}_p$ is feasible and have an interior. Then, we have $\bar{\mathbf{x}} \in \text{int } K$ and $\bar{\tau} > 0$ such that $\mathcal{A}\bar{\mathbf{x}} - \mathbf{b}\bar{\tau} = \mathbf{0}, \ (\bar{\mathbf{x}}, \bar{\tau}) \in \text{int}(K \times R_+)$. Then, for any z^*, we again consider the alternative system pair

$$\mathcal{A}\mathbf{x} - \mathbf{b}\tau = \mathbf{0}, \ \mathbf{c} \bullet \mathbf{x} - z^*\tau < 0, \ (\mathbf{x}, \tau) \in K \times R_+,$$

and

$$\mathcal{A}^T\mathbf{y} + \mathbf{s} = \mathbf{c}, \ -\mathbf{b}^T\mathbf{y} + s = -z^*, \ (\mathbf{s}, s) \in K^* \times R_+.$$

But the latter is infeasible, so that the formal has a feasible solution for any z^*. At such an solution, if $\tau > 0$, we have $\mathbf{c} \bullet (\mathbf{x}/\tau) < z^*$; if $\tau = 0$, we have $\hat{\mathbf{x}} + \alpha\mathbf{x}$, where $\hat{\mathbf{x}}$ is any feasible solution for (CLP), being feasible for (CLP) and its objective value goes to $-\infty$ as α goes to ∞.

The SDP Primal and Dual Relation

Dual \ Primal	F-B	F-UB	IF
F-B	☺		☹
F-UB			☹
IF	☹	☹	☹

$$
\begin{array}{lll}
\min & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \bullet X & \\
\text{s.t.} & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \bullet X & = 0 \\
& \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \bullet X & = 2 \\
& X & \succeq \mathbf{0}
\end{array}
$$

$$
\begin{array}{lll}
\max & 2y_2 & \\
\text{s.t.} & y_1 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + y_2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + S & = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\
& S & \succeq \mathbf{0}
\end{array}
$$

The Dual is feasible and bounded, but Primal is infeasible.

Rules to Construct the Dual in General

$$(CLP) \quad \text{minimize} \quad \sum_k \mathbf{c}_k \bullet \mathbf{x}_k$$
$$\text{subject to} \quad \sum_k \mathcal{A}_k \mathbf{x}_k = \mathbf{b},$$
$$\mathbf{x}_k \in K_k, \ \forall k.$$

$$(CLD) \quad \text{minimize} \quad \mathbf{b}^T \mathbf{y}$$
$$\text{subject to} \quad \mathcal{A}_k^T \mathbf{y} + \mathbf{s}_k = \mathbf{c}_k, \ \forall k,$$
$$\mathbf{s}_k \in K_k^*, \ \forall k.$$

obj. coef. vector	right-hand-side
right-hand-side	obj. coef. vector
$\mathcal{A}$	$\mathcal{A}^T$
Max model	Min model
$\mathbf{x}_k \in K$	kth block-constraint slack $\mathbf{s}_k \in K^*$
$\mathbf{x}_k$ "free"	kth block-constraint slack $\mathbf{s}_k = \mathbf{0}$
ith block-constraint slack $\mathbf{s}_i \in K$	$\mathbf{y}_i \in K^*$
ith block-constraint slack $\mathbf{s}_i = \mathbf{0}$	$\mathbf{y}_i$ "free"

The dual of the dual is primal!

Optimality and Complementarity Conditions for SDP

$$\begin{array}{rcl} \mathbf{c} \bullet X - \mathbf{b}^T \mathbf{y} & = & 0 \\ \mathcal{A}X & = & \mathbf{b} \\ -\mathcal{A}^T \mathbf{y} - S & = & -\mathbf{c} \\ X, S & \succeq & \mathbf{0} \end{array}, \qquad (1)$$

$$\begin{array}{rcl} XS & = & \mathbf{0} \\ \mathcal{A}X & = & \mathbf{b} \\ -\mathcal{A}^T \mathbf{y} - S & = & -\mathbf{c} \\ X, S & \succeq & \mathbf{0} \end{array} \qquad (2)$$

LP, SOCP, and SDP Examples

$$\begin{array}{llll} \text{min} & 2x_1 + x_2 + x_3 & \text{max} & y \\ \text{s. t.} & x_1 + x_2 + x_3 = 1, & \text{s.t.} & \mathbf{e} \cdot y + \mathbf{s} = (2;\ 1;\ 1), \\ & (x_1; x_2; x_3) \geq \mathbf{0}. & & (s_1; s_2; s_3) \geq \mathbf{0}. \end{array}$$

$$\begin{array}{llll} \text{min} & 2x_1 + x_2 + x_3 & \text{max} & y \\ \text{s.t.} & x_1 + x_2 + x_3 = 1, & \text{s.t.} & \mathbf{e} \cdot y + \mathbf{s} = (2;\ 1;\ 1), \\ & x_1 - \sqrt{x_2^2 + x_3^2} \geq 0. & & s_1 - \sqrt{s_2^2 + s_3^2} \geq 0. \end{array}$$

For the SOCP case: $2 - y \geq \sqrt{2(1-y)^2}$. Since $y = 1$ is feasible for the dual, $y^* \geq 1$ so that the dual constraint becomes $2 - y \geq \sqrt{2}(y-1)$ or $y \leq \sqrt{2}$. Thus, $y^* = \sqrt{2}$, and there is no duality gap.

$$
\begin{array}{ll}
\text{minimize} & \begin{pmatrix} 2 & .5 \\ .5 & 1 \end{pmatrix} \bullet \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} \\
\text{subject to} & \begin{pmatrix} 1 & .5 \\ .5 & 1 \end{pmatrix} \bullet \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} = 1, \\
& \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} \succeq \mathbf{0},
\end{array}
$$

$$
\begin{array}{ll}
\text{maximize} & y \\
\text{subject to} & \begin{pmatrix} 1 & .5 \\ .5 & 1 \end{pmatrix} y + \mathbf{s} = \begin{pmatrix} 2 & .5 \\ .5 & 1 \end{pmatrix}, \\
& \mathbf{s} = \begin{pmatrix} s_1 & s_2 \\ s_2 & s_3 \end{pmatrix} \succeq \mathbf{0}.
\end{array}
$$

Equivalence of Convex Optimization and CLP

The convex program can be rewritten as

$$
\begin{array}{lll}
(CO) & \text{minimize} & \alpha \\
 & \text{subject to} & c_0(\mathbf{x}) - \alpha \le 0, \\
 & & c_i(\mathbf{x}) \le 0, i = 1, 2, ..., m.
\end{array}
$$

Thus, it is sufficient to consider convex optimization in a form

$$
\begin{array}{lll}
(CO) & \text{minimize} & \mathbf{c}^T\mathbf{x} \\
 & \text{subject to} & c_i(\mathbf{x}) \le 0, i = 1, 2, ..., m,
\end{array}
$$

where $c_i(\mathbf{x})$, $i = 1, ..., m$, are convex functions of $\mathbf{x}$.

Convex Optimization and CLP continued

Consider set

$$\{(\tau; \mathbf{x}) : \ \tau > 0, \ \tau c_i(\mathbf{x}/\tau) \leq 0, \}$$

and K_i be its closure. Then, it is a closed and pointed convex cone !

Then, (CO) can be written as

$$\begin{array}{ll} \text{minimize} & (0; \mathbf{c}) \bullet (\tau; \mathbf{x}) \\ \text{subject to} & (1; \mathbf{0}) \bullet (\tau; \mathbf{x}) = 1, \\ & (\tau; \mathbf{x}) \in K = K_1 \cap, ..., \cap K_m, \end{array}$$

How to Construct the Dual Cone

The dual cone is the set of all points $(\kappa; \mathbf{s})$ such that

$$\kappa\tau + \mathbf{s}^T\mathbf{x} \geq 0,\ \forall(\tau; \mathbf{x}) \text{ s.t. } \tau > 0,\ \tau c_i(\mathbf{x}/\tau) \leq 0,\ i = 1, ..., m.$$

Without loss of generality, we can set $\tau = 1$ and the condition becomes

$$\kappa + \mathbf{s}^T\mathbf{x} \geq 0,\ \forall\mathbf{x} \text{ s.t. } c_i(\mathbf{x}) \leq 0,\ i = 1, ..., m.$$

Then, consider the optimization problem

$$\begin{array}{rll} \psi(\mathbf{s}) := & \inf & \mathbf{s}^T\mathbf{x} \\ & \text{s.t.} & c_i(\mathbf{x}) \leq 0, i = 1, 2, ..., m, \end{array}$$

Then, the dual cone can be represented as

$$K^* = \{(\kappa; \mathbf{s}) :\ \kappa + \psi(\mathbf{s}) \geq 0\}.$$

Example: Ellipsoidal Cone and its Dual

Let convex function $c_1(\mathbf{x}) = \sqrt{\mathbf{x}^T Q \mathbf{x}} - 1$, where data matrix Q is PD. Then $\tau c_1(\mathbf{x}/\tau) = \sqrt{\mathbf{x}^T Q \mathbf{x}} - \tau$, and $\{(\tau; \mathbf{x}) :\ \tau > 0,\ \tau c_i(\mathbf{x}/\tau) \leq 0, \}$ is called the ellipsoidal cone. If Q is an identify matrix, it reduces to the SOCP cone.

To find the dual of the cone, we consider the optimization problem

$$\begin{array}{rll} \psi(\mathbf{s}) := & \inf & \mathbf{s}^T \mathbf{x} \\ & \text{s.t.} & \sqrt{\mathbf{x}^T Q \mathbf{x}} - 1 \leq 0, \quad \text{or} \end{array}$$

$$\begin{array}{rll} \psi(\mathbf{s}) := & \inf & \mathbf{s}^T \mathbf{x} \\ & \text{s.t.} & \mathbf{x}^T Q \mathbf{x} \leq 1. \end{array}$$

The problem has a close form minimizer $\mathbf{x} = -Q^{-1}\mathbf{s}/\|Q^{-1/2}\mathbf{s}\|$ so that $\psi(\mathbf{s}) = -\sqrt{\mathbf{s}^T Q^{-1} \mathbf{s}}$, and the dual cone can be represented as

$$\{(\kappa; \mathbf{s}) :\ \kappa - \sqrt{\mathbf{s}^T Q^{-1} \mathbf{s}} \geq 0\}.$$

Duality Application: Robust Optimization

Consider a linear program

$$\begin{array}{ll} \text{minimize} & (\mathbf{c}+C\mathbf{u})^T\mathbf{x} \\ \text{subject to} & A\mathbf{x} \; = \; \mathbf{b}, \\ & \mathbf{x} \; \geq \; \mathbf{0}, \end{array}$$

where $\mathbf{u} \geq \mathbf{0}$ and $\mathbf{u} \leq \mathbf{e}$ is chosen by an Adversary and beyond decision maker's control.

Robust Min-Max Model:

$$\begin{array}{ll} \text{minimize} & \max_{\{\mathbf{u}\geq\mathbf{0},\ \mathbf{u}\leq\mathbf{e}\}}(\mathbf{c}+C\mathbf{u})^T\mathbf{x} \\ \text{subject to} & A\mathbf{x} \; = \; \mathbf{b}, \\ & \mathbf{x} \; \geq \; \mathbf{0}. \end{array}$$

The Dual of Adversary's Problem

Adversary's (primal) problem:

$$\begin{array}{rrcl} \text{maximize}_{\mathbf{u}} & \mathbf{c}^T\mathbf{x} + \mathbf{x}^T C\mathbf{u} & & \\ \text{subject to} & \mathbf{u} & \leq & \mathbf{e}, \\ & \mathbf{u} & \geq & \mathbf{0}. \end{array}$$

Dual of Adversary's problem:

$$\begin{array}{rrcl} \text{minimize}_{\mathbf{y}} & \mathbf{c}^T\mathbf{x} + \mathbf{e}^T\mathbf{y} & & \\ \text{subject to} & \mathbf{y} & \geq & C^T\mathbf{x}, \\ & \mathbf{y} & \geq & \mathbf{0}. \end{array}$$

Decision Maker's Robust Optimization Model

Robust Min-Min Model:

$$\begin{array}{ll} \text{minimize}_{\mathbf{x}} & \min_{\mathbf{y}} \ \mathbf{c}^T\mathbf{x} + \mathbf{e}^T\mathbf{y} \\ & \text{s.t.} \quad \mathbf{y} \geq C^T\mathbf{x}, \ \mathbf{y} \geq \mathbf{0} \\ \text{subject to} & A\mathbf{x} = \mathbf{b}, \\ & \mathbf{x}, \ \mathbf{y} \geq \mathbf{0}. \end{array}$$

which is equivalent to

$$\begin{array}{ll} \text{minimize}_{\mathbf{x},\mathbf{y}} & \mathbf{c}^T\mathbf{x} + \mathbf{e}^T\mathbf{y} \\ \text{subject to} & \mathbf{y} \geq C^T\mathbf{x}, \\ & A\mathbf{x} = \mathbf{b}, \\ & \mathbf{x}, \ \mathbf{y} \geq \mathbf{0}. \end{array}$$

(Distributionally) Robust Deep-Learning I

Figure 2: Result of the DRO Learning I: Original

(Distributionally) Robust Deep-Learning II

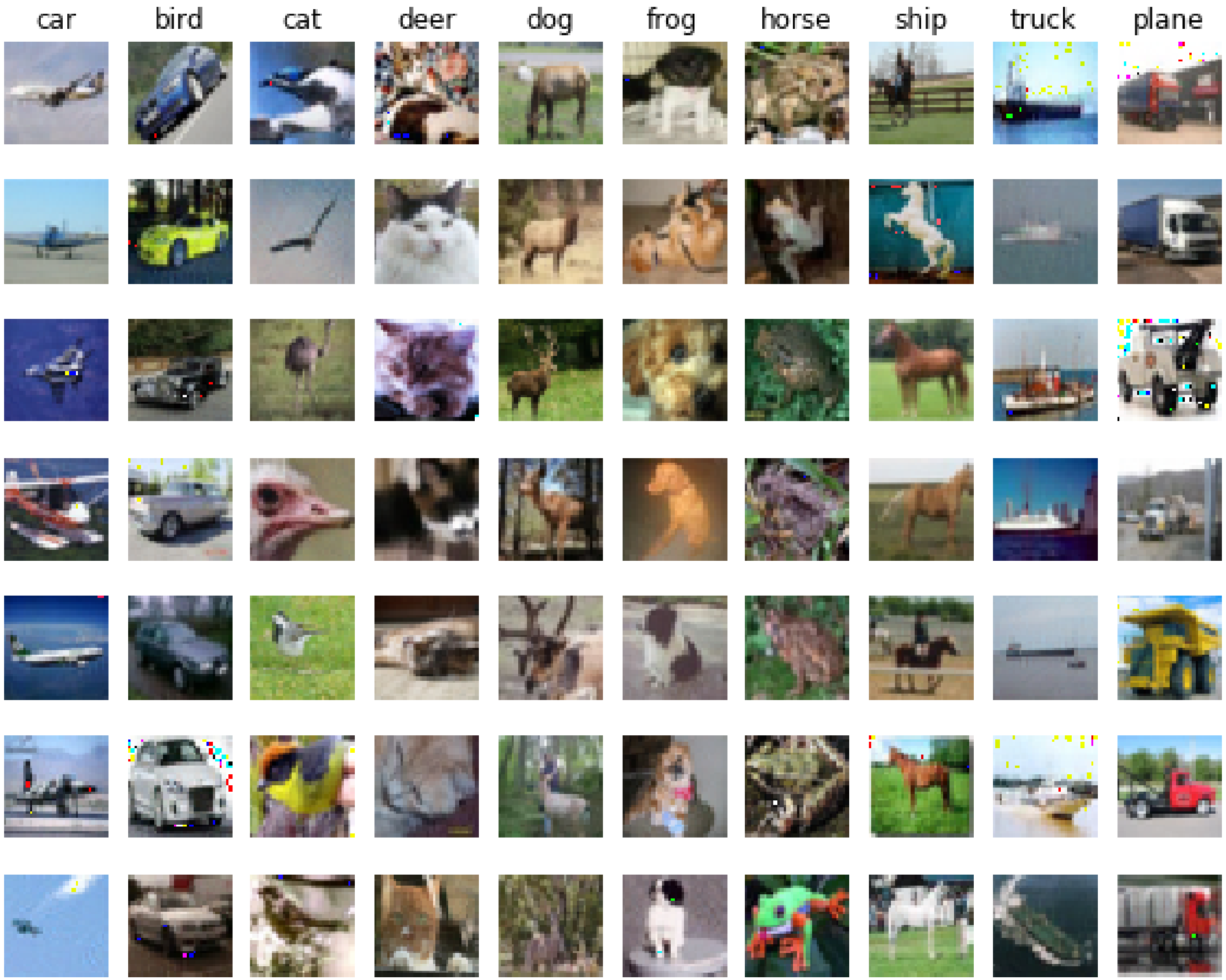

Figure 3: Result of the DRO Learning II: Nonrobust Result

(Distributionally) Robust Deep-Learning III

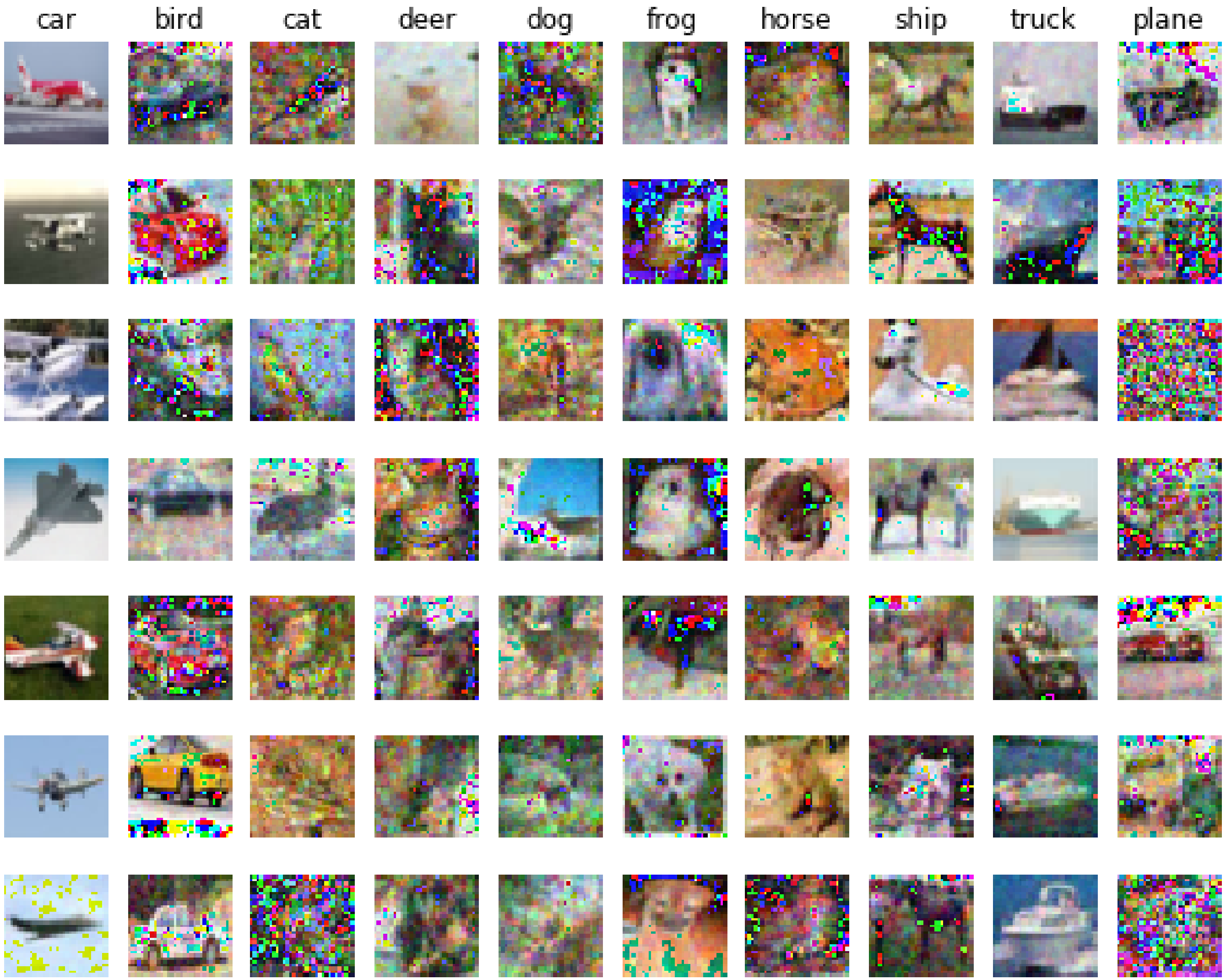

Figure 4: Result of the DRO Learning III: DRO Result

Duality Application: Combinatorial Auction Pricing I

Given the m different states that are mutually exclusive and exactly one of them will be true at the maturity. A contract on a state is a paper agreement so that on maturity it is worth a notional $\$1$ if it is on the winning state and worth $\$0$ if is not on the winning state. There are n orders betting on one or a combination of states, with a price limit and a quantity limit.

Combinatorial Auction Pricing II: an order

The jth order is given as $(\mathbf{a}_j \in R^m_+,\ \pi_j \in R_+,\ q_j \in R_+)$: $\mathbf{a}_j$ is the combination betting vector where each component is either 1 or 0

$$\mathbf{a}_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \dots \\ a_{mj} \end{pmatrix},$$

where 1 is winning and 0 is non-winning; π_j is the price limit for one such a contract, and q_j is the maximum number of contracts the better like to buy.

World Cup Information Market

Order:	$\#1$	$\#2$	$\#3$	$\#4$	$\#5$
Argentina	1	0	1	1	0
Brazil	1	0	0	1	1
Italy	1	0	1	1	0
Germany	0	1	0	1	1
France	0	0	1	0	0
Bidding Prize:π	0.75	0.35	0.4	0.95	0.75
Quantity limit:$\mathbf{q}$	10	5	10	10	5
Order fill:$\mathbf{x}$	x_1	x_2	x_3	x_4	x_5

Combinatorial Auction Pricing III: Pricing each state

Let x_j be the number of contracts awarded to the jth order. Then, the jth better will pay the amount

$$\pi_j \cdot x_j$$

and the total collected amount is

$$\sum_{j=1}^{n} \pi_j \cdot x_j = \pi^T \mathbf{x}$$

If the ith state is the winning state, then the auction organizer need to pay back

$$\left(\sum_{j=1}^{n} a_{ij} x_j \right)$$

The question is, how to decide $\mathbf{x} \in R^n$.

Combinatorial Auction Pricing IV: LP model

$$
\begin{array}{rrl}
\max & \pi^T \mathbf{x} - z & \\
\text{s.t.} & A\mathbf{x} - \mathbf{e} \cdot z & \leq \mathbf{0}, \\
& \mathbf{x} & \leq \mathbf{q}, \\
& \mathbf{x} & \geq 0.
\end{array}
$$

$\pi^T \mathbf{x}$: the optimistic amount can be collected. z: the worst-case amount need to pay back.

Combinatorial Auction V: The dual

$$\begin{array}{rrl} \min & \mathbf{q}^T\mathbf{y} & \\ \text{s.t.} & A^T\mathbf{p} + \mathbf{y} & \geq \pi, \\ & \mathbf{e}^T\mathbf{p} & = 1, \\ & (\mathbf{p}, \mathbf{y}) & \geq 0. \end{array}$$

$\mathbf{p}$ represents the state price.

What is $\mathbf{y}$?

Price information gaps/differentials/slacks where their weighted sum we like to minimize.

Combinatorial Auction V: (Strict) Complementarity

$x_j > 0$	$\mathbf{a}_j^T\mathbf{p} + y_j = \pi_j$ and $y_j \geq 0$ so that $\mathbf{a}_j^T\mathbf{p} \leq \pi_j$
$0 < x_j < q_j$	$y_j = 0$ so that $\mathbf{a}_j^T\mathbf{p} = \pi_j$
$x_j = q_j$	$y_j > 0$ so that $\mathbf{a}_j^T\mathbf{p} < \pi_j$
$x_j = 0$	$\mathbf{a}_j^T\mathbf{p} + y_j > \pi_j$ and $y_j = 0$ so that $\mathbf{a}_j^T\mathbf{p} > \pi_j$

The price is Fair:

$$\mathbf{p}^T(A\mathbf{x} - \mathbf{e}\cdot z) = 0 \quad \text{implies} \quad \mathbf{p}^T A\mathbf{x} = \mathbf{p}^T\mathbf{e}\cdot z = z;$$

that is, the worst case cost equals the worth of total shares. Moreover, if a lower bid wins the auction, so does the higher bid on any same type of bids.

World Cup Information Market Result

Order:	$\#1$	$\#2$	$\#3$	$\#4$	$\#5$	State Price
Argentina	1	0	1	1	0	0.2
Brazil	1	0	0	1	1	0.35
Italy	1	0	1	1	0	0.2
Germany	0	1	0	1	1	0.25
France	0	0	1	0	0	0
Bidding Price:π	0.75	0.35	0.4	0.95	0.75	
Quantity limit:$\mathbf{q}$	10	5	10	10	5	
Order fill:$\mathbf{x}^*$	5	5	5	0	5	

Question 1: The uniqueness of dual prices?

Combinatorial Auction Pricing VI: Convex Programming Model

$$
\begin{aligned}
\max \quad & \pi^T \mathbf{x} - z + u(\mathbf{s}) \\
\text{s.t.} \quad & A\mathbf{x} - \mathbf{e} \cdot z + \mathbf{s} = \mathbf{0}, \\
& \mathbf{x} \leq \mathbf{q}, \\
& \mathbf{x},\ \mathbf{s} \geq 0.
\end{aligned}
$$

$u(\mathbf{s})$: a value function for the market organizer on slack shares.

If $u(\cdot)$ is a strictly concave function, then the state price vector is unique.

Question 2: Online allocation?

Duality Application: Online Linear Programming

$$\begin{array}{lll} \text{maximize}_{\mathbf{x}} & \sum_{t=1}^{n} \pi_t x_t & \\ \text{subject to} & \sum_{t=1}^{n} a_{it} x_t \le b_i, & \forall i = 1, ..., m \\ & 0 \le x_t \le 1, & \forall t = 1, ..., n \end{array}$$

Each bid/activity t requests a bundle of m resources, and the payment is π_t.

Online Decision Making: we only know $(n, \mathbf{b})$ at the start, but

- the (bounded) order-data of each variable x_t is revealed sequentially.
- an irrevocable decision must be made as soon as an order arrives without observing or knowing the future data.

The algorithm/mechanism quality is evaluated on the expected performance over all the permutations comparing to the offline optimal solution, i.e., an algorithm $\mathcal{A}$ is c-competitive if and only if

$$E_\sigma \left[\sum_{t=1}^{n} \pi_t x_t(\sigma, \mathcal{A}) \right] \ge c \cdot OPT(A, \pi), \ \forall (A, \pi).$$

An Example

	order 1($t = 1$)	order 2($t = 2$)	.....	Inventory($\mathbf{b}$)
Price(π_t)	\$100	\$30	...	
Decision	x_1	x_2	...	
Pants	1	0	...	100
Shoes	1	0	...	50
T-shirts	0	1	...	500
Jacket	0	0	...	200
Socks	1	1	...	1000

Price Observation of Online Learning I

The problem would be easy if there is ”ideal price” vector:

	Bid 1($t = 1$)	Bid 2($t = 2$)	.....	Inventory($\mathbf{b}$)	$\mathbf{p}^*$
Bid(π_t)	\$100	\$30	...		
Decision	x_1	x_2	...		
Pants	1	0	...	100	\$45
Shoes	1	0	...	50	\$45
T-shirts	0	1	...	500	\$10
Jackets	0	0	...	200	\$55
Hats	1	1	...	1000	\$15

Then a simple pricing rule can be applied...

Such itemized prices exist from the offline LP shadow/dual price theorem.

Online Learning Algorithm

- Set $x_t = 0$ for all $1 \le t \le \epsilon n$;
- Solve the ϵ portion of the problem

$$\begin{array}{lll} \text{maximize}_{\mathbf{x}} & \sum_{t=1}^{\epsilon n} \pi_t x_t & \\ \text{subject to} & \sum_{t=1}^{\epsilon n} a_{it} x_t \le \epsilon b_i & i = 1, ..., m \\ & 0 \le x_t \le 1 & t = 1, ..., \epsilon n \end{array}$$

  and get the optimal dual solution $\hat{\mathbf{p}}$ of the sample LP;

- Determine the future allocation x_t as:

$$x_t = \begin{cases} 0 & \text{if } \pi_t \le \hat{\mathbf{p}}^T \mathbf{a}_t \\ 1 & \text{if } \pi_t > \hat{\mathbf{p}}^T \mathbf{a}_t \end{cases}$$

  as long as $a_{it} x_t \le b_i - \sum_{j=1}^{t-1} a_{ij} x_j$ for all i; otherwise, set $x_t = 0$.

Online Learning: Periodically resolve the sample LP with all arrived orders and update the “ideal” prices...

Main Theorems

Theorem 4 *For any fixed* $0 < \epsilon < 1$*, there is no online algorithm for solving the linear program with competitive ratio* $1-\epsilon$ *if*

$$B < \frac{\log(m)}{\epsilon^2}.$$

Theorem 5 *For any fixed* $0 < \epsilon < 1$*, there is a* $1-\epsilon$ *competitive online algorithm for solving the linear program if*

$$B \geq \Omega\left(\frac{m\log\left(n/\epsilon\right)}{\epsilon^2}\right).$$

Agrawal, Wang and Y [2010, Operations Research 2014], where the “gap” was eventually filled...