Mathematical Preliminaries

Yinyu Ye
Department of Management Science and Engineering
Stanford University
Stanford, CA 94305, U.S.A.

http://www.stanford.edu/~yyye

Appendices A, B, and C, Chapter 1

Mathematical Optimization/Programming (MP)

The class of mathematical optimization/programming problems considered in this course can all be expressed in the form

$$\begin{array}{lll} \text{(P)} & \text{minimize} & f(\mathbf{x}) \\ & \text{subject to} & \mathbf{x} \in \mathcal{X} \end{array}$$

where $\mathcal{X}$ usually specified by constraints:

$$\begin{array}{rcll} c_i(\mathbf{x}) & = & 0 & i \in \mathcal{E} \\ c_i(\mathbf{x}) & \leq & 0 & i \in \mathcal{I}. \end{array}$$

Global and Local Optimizers

A global minimizer for (P) is a vector $\mathbf{x}^*$ such that

$$\mathbf{x}^* \in \mathcal{X} \quad \text{and} \quad f(\mathbf{x}^*) \leq f(\mathbf{x}) \quad \forall \mathbf{x} \in \mathcal{X}.$$

Sometimes one has to settle for a local minimizer, that is, a vector $\bar{\mathbf{x}}$ such that

$$\bar{\mathbf{x}} \in \mathcal{X} \quad \text{and} \quad f(\bar{\mathbf{x}}) \leq f(\mathbf{x}) \quad \forall \mathbf{x} \in \mathcal{X} \cap N(\bar{x})$$

where $N(\bar{\mathbf{x}})$ is a neighborhood of $\bar{\mathbf{x}}$. Typically, $N(\bar{\mathbf{x}}) = B_\delta(\bar{\mathbf{x}})$, an open ball centered at $\bar{\mathbf{x}}$ having suitably small radius $\delta > 0$.

The value of the objective function f at a global minimizer or a local minimizer is also of interest. We call it the global minimum value or a local minimum value, respectively.

Important Terms

- decision variable/activity, data/parameter
- objective/goal/target
- constraint/limitation/requirement
- satisfied/violated
- feasible/allowable solutions
- optimal (feasible) solutions
- optimal value

Size and Complexity of Problems

- number of decision variables
- number of constraints
- bit size/number required to store the problem input data
- problem difficulty or complexity number
- algorithm complexity or convergence speed

Real n-Space; Euclidean Space

- $\mathcal{R}, \quad \mathcal{R}_+, \quad \text{int}\,\mathcal{R}_+$
- $\mathcal{R}^n, \quad \mathcal{R}^n_+, \quad \text{int}\,\mathcal{R}^n_+$
- $\mathbf{x} \geq \mathbf{y}$ means $x_j \geq y_j$ for $j = 1, 2, ..., n$
- $\mathbf{0}$: all zero vector; and $\mathbf{e}$: all one vector
- Column vector:

$$\mathbf{x} = (x_1; x_2; \ldots; x_n)$$

  and row vector:

$$\mathbf{x} = (x_1, x_2, \ldots, x_n)$$

- Inner-Product:

$$\mathbf{x} \bullet \mathbf{y} := \mathbf{x}^T\mathbf{y} = \sum_{j=1}^{n} x_j y_j$$

- Vector norm: $\|\mathbf{x}\|_2 = \sqrt{\mathbf{x}^T\mathbf{x}}$, $\|\mathbf{x}\|_\infty = \max\{|x_1|, |x_2|, ..., |x_n|\}$, in general, for $p \geq 1$

$$\|\mathbf{x}\|_p = \left(\sum_{j=1}^{n} |x_j|^p \right)^{1/p}$$

  (Quasi-norm when $0 < p < 1$.)

- A set of vectors $\mathbf{a}_1, ..., \mathbf{a}_m$ is said to be linearly dependent if there are multipliers $\lambda_1, ..., \lambda_m$, not all zero, the linear combination

$$\sum_{i=1}^{m} \lambda_i \mathbf{a}_i = \mathbf{0}$$

- A linearly independent set of vectors that span R^n is a basis.

- For a sequence $\mathbf{x}^k \in R^n$, $k = 0, 1, ...$, we say it is a contraction sequence if there is an $\mathbf{x}^* \in R^n$ and a scalar constant $0 < \gamma < 1$ such that

$$\|\mathbf{x}^{k+1} - \mathbf{x}^*\| \leq \gamma \|\mathbf{x}^k - \mathbf{x}^*\|, \ \forall k \geq 0.$$

Matrices

- $A \in \mathcal{R}^{m \times n}$; $\mathbf{a}_{i.}$, the ith row vector; $\mathbf{a}_{.j}$, the jth column vector; a_{ij}, the i, jth entry
- $\mathbf{0}$: all zero matrix, and I: the identity matrix
- The null space $\mathcal{N}(A)$, the row space $\mathcal{R}(A^T)$, and they are orthogonal.
- $\det(A)$, tr(A): the sum of the diagonal entries of A
- Inner Product:

$$A \bullet B = \text{tr} A^T B = \sum_{i,j} a_{ij} b_{ij}$$

- The operator norm of matrix A:

$$\|A\|^2 := \max_{\mathbf{0} \neq \mathbf{x} \in \mathcal{R}^n} \frac{\|A\mathbf{x}\|^2}{\|\mathbf{x}\|^2}$$

  The Frobenius norm of matrix A:

$$\|A\|_f^2 := A \bullet A = \sum_{i,j} a_{ij}^2$$

- Sometimes we use $X = \text{diag}(\mathbf{x})$
- Eigenvalues and eigenvectors

$$A\mathbf{v} = \lambda \cdot \mathbf{v}$$

- Perron-Frobenius Theorem: a real square matrix with positive entries has a unique largest real eigenvalue and that the corresponding eigenvector can be chosen to have strictly positive components.
- Stochastic Matrices: $A \geq \mathbf{0}$ with $\mathbf{e}^T A = \mathbf{e}^T$ (Column-Stochastic), or $A\mathbf{e} = \mathbf{e}$ (Row-Stochastic), or Doubly-Stochastic if both. It has a unique largest real eigenvalue 1 and corresponding non-negative right or left eigenvector.

Symmetric Matrices

- $\mathcal{S}^n$
- The Frobenius norm:

$$\|X\|_f = \sqrt{\text{tr} X^T X} = \sqrt{X \bullet X}$$

- Positive Definite (PD): $Q \succ \mathbf{0}$ iff $\mathbf{x}^T Q \mathbf{x} > 0, \quad \text{for all} \quad \mathbf{x} \neq \mathbf{0}$. The sum of PD matrices is PD.
- Positive Semidefinite (PSD): $Q \succeq \mathbf{0}$ iff $\mathbf{x}^T Q \mathbf{x} \geq 0, \quad \text{for all} \quad \mathbf{x}$. The sum of PSD matrices is PSD.
- PSD matrices: $\mathcal{S}^n_+, \quad \text{int}\, \mathcal{S}^n_+$ is the set of all positive definite matrices.

Affine Set

$S \subset R^n$ is affine if

$$[\mathbf{x}, \mathbf{y} \in S \text{ and } \alpha \in R] \Longrightarrow \alpha\mathbf{x} + (1-\alpha)\mathbf{y} \in S.$$

When $\mathbf{x}$ and $\mathbf{y}$ are two distinct points in R^n and α runs over R ,

$$\{\mathbf{z} : \mathbf{z} = \alpha\mathbf{x} + (1-\alpha)\mathbf{y}\}$$

is the affine combination of $\mathbf{x}$ and $\mathbf{y}$.

When $0 \le \alpha \le 1$, it is called the convex combination of $\mathbf{x}$ and $\mathbf{y}$. More points?

For multipliers $\alpha \ge 0$ and for $\beta \ge 0$

$$\{\mathbf{z} : \mathbf{z} = \alpha\mathbf{x} + \beta\mathbf{y}\},$$

is called the conic combination of $\mathbf{x}$ and $\mathbf{y}$.

It is called linear combination if both α and β are “free”.

Convex Set

- Ω is said to be a convex set if for every $\mathbf{x}^1, \mathbf{x}^2 \in \Omega$ and every real number $\alpha \in [0, 1]$, the point $\alpha\mathbf{x}^1 + (1-\alpha)\mathbf{x}^2 \in \Omega$.

- Ball and Ellipsoid: for given $\mathbf{y} \in \mathcal{R}^n$ and positive definite matrix Q: $E(\mathbf{y}, Q) = \{\mathbf{x} : \ (\mathbf{x}-\mathbf{y})^T Q(\mathbf{x}-\mathbf{y}) \le 1\}$.

- The intersection of convex sets is convex, the sum-set of convex sets is convex, the scaled-set of a convext set is convex

- The convex hull of a set Ω is the intersection of all convex sets containing Ω. Given column-points of A, the convex hull is $\{\mathbf{z} = A\mathbf{x} : \ \mathbf{e}^T\mathbf{x} = 1, \mathbf{x} \ge \mathbf{0}\}$.
  SVM Claim: two point sets are separable by a plane if any only if their convex hulls are separable.

- An extreme point in a convex set is a point that cannot be expressed as a convex combination of other two distinct points of the set.

- A set is polyhedral if it has finitely many extreme points; $\{\mathbf{x} : \ A\mathbf{x} = \mathbf{b}, \ \mathbf{x} \ge \mathbf{0}\}$ and $\{\mathbf{x} : \ A\mathbf{x} \le \mathbf{b}\}$ are convex polyhedral.

Cone and Convex Cone

- A set C is a cone if $\mathbf{x} \in C$ implies $\alpha\mathbf{x} \in C$ for all $\alpha > 0$
- The intersection of cones is a cone
- A convex cone is a cone and also a convex set
- A pointed cone is a cone that does not contain a line
- Dual:

$$C^* := \{\mathbf{y} : \; \mathbf{x} \bullet \mathbf{y} \geq 0 \quad \text{for all} \quad \mathbf{x} \in C\}.$$

Theorem 1 *The dual is always a closed convex cone, and the dual of the dual is the closure of convex hall of* C.

Cone Examples

- Example 1: The n-dimensional non-negative orthant, $\mathcal{R}^n_+ = \{\mathbf{x} \in \mathcal{R}^n : \ \mathbf{x} \geq \mathbf{0}\}$, is a convex cone. Its dual is itself.
- Example 2: The set of all PSD matrices in $\mathcal{S}^n$, $\mathcal{S}^n_+$, is a convex cone, called the PSD matrix cone. Its dual is itself.
- Example 3: The set $\{(t; \mathbf{x}) \in \mathcal{R}^{n+1} : \ t \geq \|\mathbf{x}\|_p\}$ for a $p \geq 1$ is a convex cone in $\mathcal{R}^{n+1}$, called the p-order cone. Its dual is the q-order cone with $\frac{1}{p} + \frac{1}{q} = 1$.
- The dual of the second-order cone ($p = 2$) is itself.

Polyhedral Convex Cones

- A cone C is (convex) polyhedral if C can be represented by

$$C = \{\mathbf{x} : A\mathbf{x} \leq \mathbf{0}\}$$

or

$$C = \{A\mathbf{x} : \mathbf{x} \geq \mathbf{0}\}$$

for some matrix A.

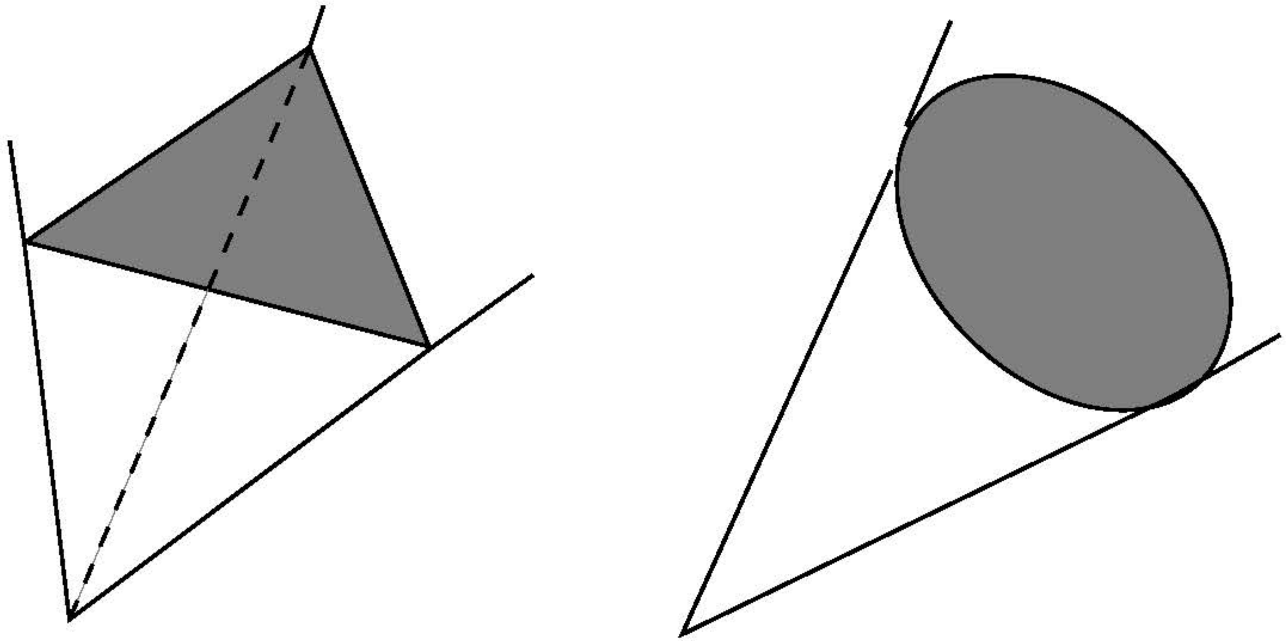

Figure 1: Polyhedral and nonpolyhedral cones.

- The non-negative orthant is a polyhedral cone, and neither the PSD matrix cone nor the second-order cone is polyhedral.

Real Functions

- Continuous functions
- Weierstrass theorem: a continuous function f defined on a compact set (bounded and closed) $\Omega \subset \mathcal{R}^n$ has a minimizer in Ω.
- The gradient vector: $\nabla f(\mathbf{x}) = \{\partial f / \partial x_i\}$, for $i = 1, ..., n.$
- The Hessian matrix: $\nabla^2 f(\mathbf{x}) = \left\{ \frac{\partial^2 f}{\partial x_i \partial x_j} \right\}$ for $i = 1, ..., n;\ j = 1, ..., n.$
- Vector function: $\mathbf{f} = (f_1; f_2; ...; f_m)$
- The Jacobian matrix of $\mathbf{f}$ is

$$\nabla \mathbf{f}(x) = \begin{pmatrix} \nabla f_1(\mathbf{x}) \\ ... \\ \nabla f_m(\mathbf{x}) \end{pmatrix}.$$

- The least upper bound or supremum of f over Ω

$$\sup\{f(\mathbf{x}) : \mathbf{x} \in \Omega\}$$

and the greatest lower bound or infimum of f over Ω

$$\inf\{f(\mathbf{x}) : \mathbf{x} \in \Omega\}$$

Convex Functions

- f is a (strongly) convex function iff for $0 < \alpha < 1$,

$$f(\alpha\mathbf{x} + (1-\alpha)\mathbf{y})(<) \leq \alpha f(\mathbf{x}) + (1-\alpha)f(\mathbf{y}).$$

- The sum of convex functions is a convex function; the max of convex functions is a convex function;
- The Composed function $f(\phi(\mathbf{x}))$ is convex if $\phi(\mathbf{x})$ is a convex and $f(\cdot)$ is convex&non-decreasing.
- The (lower) level set of f is convex:

$$L(z) = \{\mathbf{x} : \quad f(\mathbf{x}) \leq z\}.$$

- Convex set $\{(z;\mathbf{x}) : \ f(\mathbf{x}) \leq z\}$ is called the epigraph of f.
- $tf(\mathbf{x}/t)$ is a convex function of $(t;\mathbf{x})$ for $t > 0$ if $f(\cdot)$ is a convex function; it's homogeneous with degree 1.

Convex Function Examples

- $\|\mathbf{x}\|_p$ for $p \geq 1$.

$$\|\alpha\mathbf{x} + (1-\alpha)\mathbf{y}\|_p \leq \|\alpha\mathbf{x}\|_p + \|(1-\alpha)\mathbf{y}\|_p \leq \alpha\|\mathbf{x}\|_p + (1-\alpha)\|\mathbf{y}\|_p,$$

  from the triangle inequality.

- Logistic function $\log(1 + e^{\mathbf{a}^T\mathbf{x}+b})$ is convex.

- $e^{x_1} + e^{x_2} + e^{x_3}$.

- $\log(e^{x_1} + e^{x_2} + e^{x_3})$: we will prove it later.

Theorem 2 *Every local minimizer is a global minimizer in minimizing a convex objective function over a convex feasible set. If the objective is strongly convex in the feasible set, the minimizer is unique.*

Theorem 3 *Every local minimizer is a boundary solution in minimizing a concave objective function (with non-zero gradient everywhere) over a convex feasible set. If the objective is strongly concave in the feasible set, every local minimizer must be an extreme solution.*

Example: Proof of convex function

Consider the minimal-objective function of $\mathbf{b}$ for fixed A and $\mathbf{c}$:

$$
\begin{aligned}
z(\mathbf{b}) := \text{minimize} \quad & f(\mathbf{x}) \\
\text{subject to} \quad & A\mathbf{x} = \mathbf{b}, \\
& \mathbf{x} \geq \mathbf{0},
\end{aligned}
$$

where $f(\mathbf{x})$ is a convex function.

Show that $z(\mathbf{b})$ is a convex function in $\mathbf{b}$.

Theorems on functions

Taylor's theorem or the mean-value theorem:

Theorem 4 *Let $f \in C^1$ be in a region containing the line segment $[\mathbf{x}, \mathbf{y}]$. Then there is a α, $0 \le \alpha \le 1$, such that*

$$f(\mathbf{y}) = f(\mathbf{x}) + \nabla f(\alpha\mathbf{x} + (1-\alpha)\mathbf{y})(\mathbf{y} - \mathbf{x}).$$

Furthermore, if $f \in C^2$ then there is a α, $0 \le \alpha \le 1$, such that

$$f(\mathbf{y}) = f(\mathbf{x}) + \nabla f(\mathbf{x})(\mathbf{y} - \mathbf{x}) + (1/2)(\mathbf{y} - \mathbf{x})^T \nabla^2 f(\alpha\mathbf{x} + (1-\alpha)\mathbf{y})(\mathbf{y} - \mathbf{x}).$$

Theorem 5 *Let $f \in C^1$. Then f is convex over a convex set Ω if and only if*

$$f(\mathbf{y}) \ge f(\mathbf{x}) + \nabla f(\mathbf{x})(\mathbf{y} - \mathbf{x})$$

for all $\mathbf{x},\ \mathbf{y} \in \Omega$.

Theorem 6 *Let $f \in C^2$. Then f is convex over a convex set Ω if and only if the Hessian matrix of f is positive semi-definite throughout Ω.*

Theorem 7 *Suppose we have a set of m equations in n variables*

$$h_i(\mathbf{x}) = 0,\ i = 1, ..., m$$

where $h_i \in C^p$ for some $p \geq 1$. Then, a set of m variables can be expressed as implicit functions of the other $n - m$ variables in the neighborhood of a feasible point when the Jacobian matrix of the m functions is nonsingular.

Lipschitz Functions

The first-order β-Lipschitz function: there is a positive number β such that for any two points $\mathbf{x}$ and $\mathbf{y}$:

$$\|\nabla f(\mathbf{x}) - \nabla f(\mathbf{y})\| \leq \beta \|\mathbf{x} - \mathbf{y}\|. \quad (1)$$

This condition imples

$$|f(\mathbf{x}) - f(\mathbf{y}) - \nabla f(\mathbf{y})^T (\mathbf{x} - \mathbf{y})| \leq \frac{\beta}{2} \|\mathbf{x} - \mathbf{y}\|^2.$$

The second-order β-Lipschitz function: there is a positive number β such that for any two points $\mathbf{x}$ and $\mathbf{y}$

$$\|\nabla f(\mathbf{x}) - \nabla f(\mathbf{y}) - \nabla^2 f(\mathbf{y})(\mathbf{x} - \mathbf{y})\| \leq \beta \|\mathbf{x} - \mathbf{y}\|^2. \quad (2)$$

This condition implies

$$|f(\mathbf{x}) - f(\mathbf{y}) - \nabla f(\mathbf{y})^T (\mathbf{x} - \mathbf{y}) - \frac{1}{2}(\mathbf{x} - \mathbf{y})^T \nabla^2 f(\mathbf{y})(\mathbf{x} - \mathbf{y})| \leq \frac{\beta}{3} \|\mathbf{x} - \mathbf{y}\|^3.$$

Known Inequalities

- Cauchy-Schwarz: given $\mathbf{x}, \mathbf{y} \in \mathcal{R}^n$, $|\mathbf{x}^T\mathbf{y}| \leq \|\mathbf{x}\|_p \|\mathbf{y}\|_q$, where $\frac{1}{p} + \frac{1}{q} = 1$ and $p \geq 1$.
- Triangle: given $\mathbf{x}, \mathbf{y} \in \mathcal{R}^n$, $\|\mathbf{x} + \mathbf{y}\|_p \leq \|\mathbf{x}\|_p + \|\mathbf{y}\|_p$ for $p \geq 1$.
- Arithmetic-geometric mean: given $\mathbf{x} \in \mathcal{R}^n_+$,

$$\frac{\sum x_j}{n} \geq \left(\prod x_j\right)^{1/n}.$$

System of linear equations

Given $A \in \mathcal{R}^{m \times n}$ and $\mathbf{b} \in \mathcal{R}^m$, the problem is to determine n unknowns from m linear equations:

$$A\mathbf{x} = \mathbf{b}$$

Theorem 8 *Let $A \in \mathcal{R}^{m \times n}$ and $\mathbf{b} \in \mathcal{R}^m$. The system $\{\mathbf{x} : \ A\mathbf{x} = \mathbf{b}\}$ has a solution if and only if that $A^T\mathbf{y} = \mathbf{0}$ and $\mathbf{b}^T\mathbf{y} \neq 0$ has no solution.*

A vector $\mathbf{y}$, with $A^T\mathbf{y} = 0$ and $\mathbf{b}^T\mathbf{y} \neq 0$, is called an infeasibility certificate for the system.

Alternative system pairs: $\{\mathbf{x} : \ A\mathbf{x} = \mathbf{b}\}$ and $\{\mathbf{y} : \ A^T\mathbf{y} = \mathbf{0}, \ \mathbf{b}^T\mathbf{y} \neq 0\}$.

Gaussian Elimination and LU Decomposition

$$\begin{pmatrix} a_{11} & A_{1.} \\ 0 & A' \end{pmatrix} \begin{pmatrix} x_1 \\ x' \end{pmatrix} = \begin{pmatrix} b_1 \\ b' \end{pmatrix}.$$

$$A = L \begin{pmatrix} U & C \\ 0 & 0 \end{pmatrix}$$

The method runs in $O(n^3)$ time for n equations with n unknowns.

Linear least-squares problem

Given $A \in \mathcal{R}^{m\times n}$ and $\mathbf{c} \in \mathcal{R}^n$,

$$\begin{array}{lll} (LS) & \text{minimize} & \|A^T\mathbf{y} - \mathbf{c}\|^2 \\ & \text{subject to} & \mathbf{y} \in \mathcal{R}^m, \quad \text{or} \end{array}$$

$$\begin{array}{lll} (LS) & \text{minimize} & \|\mathbf{s} - \mathbf{c}\|^2 \\ & \text{subject to} & \mathbf{s} \in \mathcal{R}(A^T). \end{array}$$

$$AA^T\mathbf{y} = A\mathbf{c}$$

Choleski Decomposition:

$$AA^T = L\Lambda L^T, \quad \text{and then solve} \quad L\Lambda L^T\mathbf{y} = A\mathbf{c}.$$

Projections Matrices: $A^T(AA^T)^{-1}A$ and $I - A^T(AA^T)^{-1}A$

Solving ball-constrained linear problem

$$(BP) \quad \text{minimize} \quad \mathbf{c}^T\mathbf{x}$$
$$\text{subject to} \quad A\mathbf{x} = \mathbf{0},\ \|\mathbf{x}\|^2 \le 1,$$

$\mathbf{x}^*$ minimizes (BP) if and only if there always exists a $\mathbf{y}$ such that they satisfy

$$AA^T\mathbf{y} = A\mathbf{c},$$

and if $\mathbf{c} - A^T\mathbf{y} \neq \mathbf{0}$ then

$$\mathbf{x}^* = -(\mathbf{c} - A^T\mathbf{y})/\|\mathbf{c} - A^T\mathbf{y}\|;$$

otherwise any feasible $\mathbf{x}$ is a minimal solution.

Solving ball-constrained linear problem

$$(BD) \quad \text{minimize} \quad \mathbf{b}^T\mathbf{y}$$
$$\text{subject to} \quad \|A^T\mathbf{y}\|^2 \leq 1.$$

The solution $\mathbf{y}^*$ for (BD) is given as follows: Solve

$$AA^T\bar{\mathbf{y}} = \mathbf{b}$$

and if $\bar{\mathbf{y}} \neq \mathbf{0}$ then set

$$\mathbf{y}^* = -\bar{\mathbf{y}}/\|A^T\bar{\mathbf{y}}\|;$$

otherwise any feasible $\mathbf{y}$ is a solution.