CME 307 / MS&E 311 | Winter 2022
Optimization | Jan 18, 2022
Prof. Yinyu Ye | Not graded

Homework Assignment 1
Problem Session Friday Jan 28th 11:30 am

Optional Reading. Read Luenberger and Ye's *Linear and Nonlinear Programming Fourth Edition* Chapters 1, 2, 6 and Appendices A and B.

Theoretical Homework (80'):

1. (15') Show the following:

   (a) (5') Consider the set

   $$F := \{\mathbf{x} \in R^n : \ A\mathbf{x} = \mathbf{b}, \ \mathbf{x} \geq 0\},$$

   where data matrix $A \in R^{m \times n}$ and vector $\mathbf{b} \in R^m$. Prove that F is a convex set.

   (b) (5') Fix data matrix A and consider the $\mathbf{b}$-data set for F defined in part (a):

   $$B := \{\mathbf{b} \in R^m : \ F \text{ is not empty}\}.$$

   Prove that B is a convex set.

   (c) (5') Fix data matrix A and consider the linearly constrained convex minimization problem

   $$\begin{aligned} z(\mathbf{b}) := \max \quad & f(\mathbf{x}) \\ \text{s.t.} \quad & A\mathbf{x} = \mathbf{b}, \ \mathbf{x} \geq \mathbf{0} \end{aligned}$$

   where $f(\mathbf{x})$ is a concave function, and the maximal value function $z(\mathbf{b})$ is an implicit function of $\mathbf{b}$. Prove that $z(\mathbf{b})$ is a concave function of $\mathbf{b} \in B$, where B is defined in part (b).

2. (10') Show that the dual cone of the n-dimensional nonnegative orthant cone R^n_+ is itself, that is,

   $$(R^n_+)^* = R^n_+.$$

   (Hint: show that $R^n_+ \subset (R^n_+)^*$ and $(R^n_+)^* \subset R^n_+$.)

3. (10') Let $g_1, \dots, g_m$ be a collection of concave functions on R^n such that

$$S = \{\mathbf{x} : g_i(\mathbf{x}) > 0 \text{ for } i = 1, \dots, m\} \neq \emptyset.$$

Show that for any positive constant μ and any convex function f on R^n, the function (called Barrier function)

$$h(\mathbf{x}) = f(\mathbf{x}) - \mu \sum_{i=1}^{m} \log(g_i(\mathbf{x}))$$

is convex over S. (Hint: directly apply the convex/concave function definition or analyze the Hessian of $h(x)$.)

4. (10') (Lipschitz Functions) Prove the following two implication inequalities:

   (a) (5') Assume f is a first-order β-Lipschitz function, namely there is a positive number β such that for any $\mathbf{x}, \mathbf{y} \in R^n$:

   $$\|\nabla f(\mathbf{x}) - \nabla f(\mathbf{y})\| \leq \beta \|\mathbf{x} - \mathbf{y}\|,$$

   then for any $\mathbf{x}, \mathbf{y} \in R^n$,

   $$|f(\mathbf{x}) - f(\mathbf{y}) - \nabla f(\mathbf{y})^T (\mathbf{x} - \mathbf{y})| \leq \frac{\beta}{2} \|\mathbf{x} - \mathbf{y}\|^2.$$

   (b) (5') Assume f is a second-order β-Lipschitz function, namely there is a positive number β such that for any $\mathbf{x}, \mathbf{y} \in R^n$:

   $$\|\nabla f(\mathbf{x}) - \nabla f(\mathbf{y}) - \nabla^2 f(\mathbf{y})(\mathbf{x} - \mathbf{y})\| \leq \beta \|\mathbf{x} - \mathbf{y}\|^2,$$

   then for any $\mathbf{x}, \mathbf{y} \in R^n$,

   $$|f(\mathbf{x}) - f(\mathbf{y}) - \nabla f(\mathbf{y})^T (\mathbf{x} - \mathbf{y}) - \frac{1}{2} (\mathbf{x} - \mathbf{y})^T \nabla^2 f(\mathbf{y}) (\mathbf{x} - \mathbf{y})| \leq \frac{\beta}{3} \|\mathbf{x} - \mathbf{y}\|^3.$$

5. (10') Consider the following SOCP problem:

$$\begin{aligned} \min \quad & 2x_1 + x_2 + x_3 \\ \text{s.t.} \quad & x_1 + x_2 + x_3 = 1, \\ & x_1 - \sqrt{x_2^2 + x_3^2} \geq 0. \end{aligned}$$

   (a) (5') Show that the feasible region is a convex set.

   (b) (5') Try to find a minimizer of the problem and "argue" [1] why it is a minimizer.

[1] We recommend to prove this directly, namely without using duality argument which will be introduced in the following lectures.

6. (10') Prove that the set $\{A\mathbf{x} : \mathbf{x} \geq 0 \in R^n\}$ is a closed and convex cone. (Hint: apply Carathéodory's theorem in Lecture Note to prove the closedness.)

7. (15') Farkas' lemma can be used to derive many other (named) theorems of the alternative. This problem concerns a few of these pairs of systems. Using Farkas's lemma, prove each of the following results.

   (a) (5') Gordan's Theorem. Exactly one of the following systems has a solution:

$$\text{(i)} \quad A\mathbf{x} > 0$$
$$\text{(ii)} \quad \mathbf{y}^T A = 0, \quad \mathbf{y} \geq 0, \quad \mathbf{y} \neq 0.$$

   (b) (5') Stiemke's Theorem. Exactly one of the following systems has a solution:

$$\text{(i)} \quad A\mathbf{x} \geq 0, \quad A\mathbf{x} \neq 0$$
$$\text{(ii)} \quad \mathbf{y}^T A = 0, \quad \mathbf{y} > 0$$

   (c) (5') Gale's Theorem. Exactly one of the following systems has a solution:

$$\text{(i)} \quad A\mathbf{x} \leq \mathbf{b}$$
$$\text{(ii)} \quad \mathbf{y}^T A = 0, \quad \mathbf{y}^T \mathbf{b} < 0, \quad \mathbf{y} \geq 0$$

Computational Homework (50') (group of 1-3 people):

8. (20') Consider the sensor localization problem on plane R^2 with one sensor $\mathbf{x}$ and three anchors $\mathbf{a}_1 = (1;0)$, $\mathbf{a}_2 = (-1;0)$ and $\mathbf{a}_3 = (0;2)$. Suppose the Euclidean distances from the sensor to the three anchors are d_1, d_2 and d_3 respectively and known to us.Then, from the anchor and distance information, we can locate the second by finding $\mathbf{x} \in R^2$ such that

$$\|\mathbf{x} - \mathbf{a}_i\|^2 = d_i^2,\ i = 1,2,3.$$

Do the following numerical experiments using CVX (or cvxpy, convex.jl) or MOSEK and answer the questions:

(a) (10') Generate any sensor point **in the convex hull** of the three anchors, compute its distances to three anchors d_i, $i = 1,2,3$, respectively. Then solve the SOCP relaxation problem

$$\|\mathbf{x} - \mathbf{a}_i\|^2 \le d_i^2,\ i = 1,2,3.$$

Did you find the correct location? What about if the sensor point was in the **outside** of the convex hull? Try a few different locations of the sensor and identify the pattern.

(b) (10') Now try the SDP relaxation

$$(\mathbf{a}_i;-1)(\mathbf{a}_i;-1)^T \bullet \begin{pmatrix} I & \mathbf{x} \\ \mathbf{x}^T & \mathbf{y} \end{pmatrix} = d_i^2,\ i = 1,2,3;\ \begin{pmatrix} I & \mathbf{x} \\ \mathbf{x}^T & \mathbf{y} \end{pmatrix} \succeq 0 \in S^3,$$

which can be written in the standard form

$$\begin{aligned} (1;0;0)(1;0;0)^T \bullet Z &= 1, \\ (0;1;0)(0;1;0)^T \bullet Z &= 1, \\ (1;1;0)(1;1;0)^T \bullet Z &= 2, \\ (\mathbf{a}_i;-1)(\mathbf{a}_i;-1)^T \bullet Z &= d_i^2,\ i = 1,2,3, \\ Z &\succeq 0 \in S^3. \end{aligned}$$

Did you find the correct location everywhere on the plane? Try a few different locations of the sensor and identify the pattern.

You can use CVX (or cvxpy, convex.jl) to solve these numerical problems.

9. (20') Consider the sensor localization problem on plane R^2 with **two** sensors $\mathbf{x}_1$ and $\mathbf{x}_2$ and three anchors $\mathbf{a}_1 = (1;0)$, $\mathbf{a}_2 = (-1;0)$ and $\mathbf{a}_3 = (0;2)$. Suppose that we know the (Euclidean) distances from one sensor $\mathbf{x}_1$ to $\mathbf{a}_1$ and $\mathbf{a}_2$, denoted by d_{11} and d_{12}; distances of the other sensor $\mathbf{x}_2$ to $\mathbf{a}_2$ and $\mathbf{a}_3$, denoted by d_{22} and d_{23}; and the distance between the two sensors $\mathbf{x}_1$ and $\mathbf{x}_2$, denoted by $\hat{d}_{12}$. Then, from the anchor and distance information we would like to locate the sensor positions $\mathbf{x}_1, \mathbf{x}_2 \in R^2$.

   Do the following numerical experiments using CVX (or cvxpy, convex.jl) or MOSEK and answer the questions:

   (a) (10') Generate two sensor points anywhere and try the SOCP relaxation model

   $$\begin{aligned} \|\mathbf{x}_1 - \mathbf{a}_i\|^2 &\leq d_{1i}^2, \ i = 1, 2 \\ \|\mathbf{x}_2 - \mathbf{a}_i\|^2 &\leq d_{2i}^2, \ i = 2, 3 \\ \|\mathbf{x}_1 - \mathbf{x}_2\|^2 &\leq \hat{d}_{12}^2. \end{aligned}$$

   Did you find the correct locations? What have you observed? Try a few different locations of the sensor pairs and identify the pattern.

   (b) (10') Now try the SDP relaxation: find $X = [\mathbf{x}_1, \ \mathbf{x}_2] \in R^{2\times 2}$ and

   $$Z = \begin{pmatrix} I & X \\ X^T & Y \end{pmatrix} \in S^4$$

   to meet the constraints in the standard form:

   $$\begin{aligned} (1;0;0;0)(1;0;0;0)^T \bullet Z &= 1, \\ (0;1;0;0)(0;1;0;0)^T \bullet Z &= 1, \\ (1;1;0;0)(1;1;0;0)^T \bullet Z &= 2, \\ (\mathbf{a}_i;-1;0)(\mathbf{a}_i;-1;0)^T \bullet Z &= d_{1i}^2, \ i = 1, 2, \\ (\mathbf{a}_i;0;-1)(\mathbf{a}_i;0;-1)^T \bullet Z &= d_{2i}^2, \ i = 2, 3, \\ (0;0;1;-1)(0;0;1;-1)^T \bullet Z &= \hat{d}_{12}^2, \\ Z &\succeq 0 \in S^4. \end{aligned}$$

   Did you find the correct locations? What have you observed? Can you conclude with something? Try a few different locations of the sensor pairs and identify the pattern.

10. (10') For the Maze Runner example in Lecture Note #1, suppose that the blue-action at State 3 has a probability 0.5 leading to State 4 and 0.5 leading to State 5; and the only action at State 5 leads to State 0. Reformulate the MDP-LP problem with $\gamma = 0.9$ and solve it using any LP solver.