Midterm Review

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Formulation of LP: Four-Step Rule

- Sort out data and parameters from the verbal description
- Define the set of decision variables
- Formulate the objective function of data and decision variables
- Set up equality and/or inequality constraints

$$\begin{array}{rl} \text{minimize} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & A\mathbf{x} = \mathbf{b}, \\ & \mathbf{x} \geq \mathbf{0}. \end{array}$$

Separating hyperplane theorem

The most important theorem about the convex set is the following separating theorem.

Theorem 1 *(Separating hyperplane theorem) Let $C \subset \mathcal{E}$, where $\mathcal{E}$ is either $\mathcal{R}^n$ or $\mathcal{M}^n$, be a closed convex set and let y be a point exterior to C. Then there is a vector $a \in \mathcal{E}$ such that*

$$a \bullet y < \inf_{x \in C} a \bullet x.$$

Farkas' Lemma

The following results are Farkas' lemma and its variants.

Theorem 2 *Let* $A \in \mathcal{R}^{m \times n}$ *and* $b \in \mathcal{R}^m$*. Then, the system* $\{x: \; Ax = b, \; x \geq 0\}$ *has a feasible solution* x *if and only if that* $A^T y \leq 0$ *implies* $b^T y \leq 0$*.*

A vector y, with $A^T y \leq 0$ and $b^T y = 1$, is called a (primal) infeasibility certificate for the system $\{x: \; Ax = b, \; x \geq 0\}$.

Geometrically, Farkas' lemma means that if a vector $b \in \mathcal{R}^m$ does not belong to the cone generated by $a_{.1}, ..., a_{.n}$, then there is a hyperplane separating b from cone$(a_{.1}, ..., a_{.n})$.

Example

Let $A = (1, 1)$ and $b = -1$. Then, $y = -1$ is an infeasibility certificate for $\{x: \; Ax = b, \; x \geq 0\}$.

Theorem 3 *Let $A \in \mathcal{R}^{m\times n}$ and $c \in \mathcal{R}^n$. Then, the system $\{y : \ A^T y \le c\}$ has a solution y if and only if that $Ax = 0$ and $x \ge 0$ imply $c^T x \ge 0$.*

Again, a vector $x \ge 0$, with $Ax = 0$ and $c^T x = -1$, is called a (dual) infeasibility certificate for the system $\{y : \ A^T y \le c\}$.

example

Let $A = (1; -1)$ and $c = (1; -2)$. Then, $x = (1; 1)$ is an infeasibility certificate for $\{y : \ A^T y \le c\}$.

Duality Theory

Consider the linear program in standard form, called the primal problem,

$$\begin{array}{lll} (LP) & \text{minimize} & c^T x \\ & \text{subject to} & Ax = b,\ x \geq 0, \end{array}$$

where $x \in \mathcal{R}^n$.

The dual problem can be written as:

$$\begin{array}{lll} (LD) & \text{maximize} & b^T y \\ & \text{subject to} & A^T y + s = c,\ s \geq 0, \end{array}$$

where $y \in \mathcal{R}^m$ and $s \in \mathcal{R}^n$. The components of s are called dual slacks.

Duality Theory

Theorem 4 *(Weak duality theorem) Let $\mathcal{F}_p$ and $\mathcal{F}_d$ be non-empty. Then,*

$$c^T x \geq b^T y \quad \textit{where} \quad x \in \mathcal{F}_p,\ (y, s) \in \mathcal{F}_d.$$

$$c^T x - b^T y = c^T x - (Ax)^T y = x^T (c - A^T y) = x^T s \geq 0.$$

This theorem shows that a feasible solution to either problem yields a bound on the value of the other problem. We call $c^T x - b^T y$ the duality gap.

From this we have important results in the following.

Theorem 5 *(Strong duality theorem) Let $\mathcal{F}_p$ and $\mathcal{F}_d$ be non-empty. Then, x^* is optimal for (LP) if and only if the following conditions hold:*

i) $x^* \in \mathcal{F}_p$;

ii) *there is* $(y^*, s^*) \in \mathcal{F}_d$;

iii) $c^T x^* = b^T y^*$.

Given $\mathcal{F}_p$ and $\mathcal{F}_d$ being non-empty, we like to prove that there is $x^* \in \mathcal{F}_p$ and $(y^*, s^*) \in \mathcal{F}_d$ such that $c^T x^* \leq b^T y^*$, or to prove that

$$Ax = b, \; A^T y \leq c, \; c^T x - b^T y \leq 0, \; x \geq 0$$

is feasible.

Theorem 6 *(LP duality theorem) If (LP) and (LD) both have feasible solutions then both problems have optimal solutions and the optimal objective values of the objective functions are equal.*

If one of (LP) or (LD) has no feasible solution, then the other is either unbounded or has no feasible solution. If one of (LP) or (LD) is unbounded then the other has no feasible solution.

The above theorems show that if a pair of feasible solutions can be found to the primal and dual problems with equal objective values, then these are both optimal. The converse is also true; there is no “gap.”

For feasible x and (y, s), $x^T s = x^T(c - A^T y) = c^T x - b^T y$ is called the complementarity gap.

If $x^T s = 0$, then we say x and s are complementary to each other.

Since both x and s are nonnegative, $x^T s = 0$ implies that $x_j s_j = 0$ for all $j = 1, \ldots, n$.

$$\begin{aligned} Xs &= 0 \\ Ax &= b \\ -A^T y - s &= -c. \end{aligned}$$

This system has total $2n + m$ unknowns and $2n + m$ equations including n nonlinear equations.

Rules to construct the dual

obj. coef. vector	right-hand-side
right-hand-side	obj. coef. vector
A	A^T
Max model	Min model
$x_j \geq 0$	jth constraint $\geq$
$x_j \leq 0$	jth constraint $\leq$
x_j free	jth constraint $=$
ith constraint $\leq$	$y_i \geq 0$
ith constraint $\geq$	$y_i \leq 0$
ith constraint $=$	y_i free

Basic Feasible Solution

In the LP standard form, select m linearly independent columns, denoted by the index set B, from A.

$$A_B x_B = b$$

for the m-vector x_B. By setting the variables, x_N, of x corresponding to the remaining columns of A equal to zero, we obtain a solution x such that

$$Ax = b.$$

Then, x is said to be a (primal) basic solution to (LP) with respect to the basis A_B. The components of x_B are called basic variables.

If a basic solution $x \geq 0$, then x is called a basic feasible solution.

If one or more components in x_B has value zero, that basic feasible solution x is said to be (primal) degenerate.

A dual vector y satisfying

$$A_B^T y = c_B$$

is said to be the corresponding dual basic solution.

If the dual basic solution is also feasible, that is

$$s = c - A^T y \geq 0.$$

If one or more slacks in $c_N - A_N^T y$ has value zero, that dual basic feasible solution y is said to be (primal) degenerate.

Theorem 7 *(LP fundamental theorem) Given (LP) and (LD) where A has full row rank m,*

i) *if there is a feasible solution, there is a basic feasible solution;*

ii) *if there is an optimal solution, there is an optimal basic solution.*

The Simplex Algorithm

0. Initialize with a minimization problem in feasible canonical form with respect to a basic index set B. Let N denote the complementary index set.

1. Test for termination. First find

$$e \in \text{argmin}_{j \in N}\{r_j\}.$$

If $r_e \geq 0$, stop. The solution is optimal. Otherwise determine whether the column of $\bar{A}_{.e}$ contains a positive entry. If not, the objective function is unbounded below. Terminate. Let x_e be the entering basic variable.

2. Determine the outgoing Execute the MRT to determine the outgoing variable x_o.

3. Update basis. Update B and A_B and transform the probolem in cannonical form, and return to Step 1.

The Simplex Algorithm in Tableau

0. Initialize with a minimization problem in feasible canonical form with respect to a basic index set B. Let N denote the complementary index set.

1. Test for termination. First find

$$e \in \text{argmin}_{j \in N}\{r_j\}.$$

If $r_e \geq 0$, stop. The solution is optimal. Otherwise determine whether the column of x_e contains a positive entry. If not, the objective function is unbounded below. Terminate.

2. Minimum ratio test. Execute the MRT to determine the pivot row o and the pivot element, $\bar{a}_{oe}$.

3. Pivot step. Pivot on $\bar{a}_{oe}$ and modify the definitions of B and N. Return to Step 1.

The Two-Phase Simplex Method

We know that in order to begin the Simplex Method, we need to find an initial basic feasible solution of the problem constraints (if one exists). One approach to doing this is by solving the so-called Phase I Problem. The technique uses the Simplex Method itself to solve a related problem for which a starting basic feasible solution is known and for which an optimal solution must exist. If Phase I results in the discovery of a basic feasible solution for the originally stated constraints, then we can initiate Phase II wherein the Simplex Method is applied to the solving the originally stated linear programming problem. The combination of Phases I and II gives rise to the Two-Phase Simplex Method. Since there are two different linear programs being solved in these phases, it is advantageous to have a “smooth transition” between them.

Uniqueness and Degenerary

Theorem 8 *If the LP has a non-degenerate optimal dual solution, that is, the reduced cost of every nonbasic variable is positive, then the optimal primal solution is unique; if the optimal dual solution is non-degenerate and unique, then the optimal primal solution is non-degenerate and unique.*

Theorem 9 *Let (LP) denote a linear program in standard form and let (LD) denote its dual. If one of these problems has a unique and nondegenerate optimal solution, then so does the other; and the strictly complementary partition is $|P| = m$ and $|Z| = n - m$.*

Sensitivity Analysis

Let us consider λ around 0.

A key question in these parametric problems is: how much can the parameter λ be changed before the current optimal basic solution of LP(0) is lost?

Theorem 10 *The optmal basis of LP(0) remains optimal for LP(λ) if and only if*

$$A_B^{-1}(b + \lambda d) \geq 0 \quad \textit{and} \quad (c + \lambda g) - A^T(A_B^T)^{-1}(c + \lambda g)_B \geq 0.$$

This will establish an interval on λ.

Properties of the dual (shadow) prices $\mathbf{y}$

- All inactive constraint have zero dual price from complementarity.
- In general, the dual price on a given active constraint is the rate of change in the OV as the RHS of the constraint increases, ceteris paribus.
- If the OS is degenerate, the dual price may be valid for one-sided changes in the RHS.
- The constraint RHS ranges give the ranges of the constraint RHS over which no change in the optimal basis will occur.
- One of the allowable increase and decrease for an inactive constraint is infinite and the other equals to the slack or surplus.
- In general, when the RHS of an active constraint changes, both the OV and OS will change.

Properties of the dual slacks (reduced costs) $\mathbf{r}$?

- All basic variables have zero dual slack value from complementarity.
- In general, the dual slack value of any non-basic variable is the amount the objective coefficient of that variable would have to change, with all other data held fixed, in order for it to become a basic variable at optimality.
- If the OS is degenerate, the objective coefficient of a non-basic variable would have to change by at least the slack value in order to become a basic variable.
- The objective coefficient ranges give the ranges of the objective function over which no change in the optimal basis will occur.
- One of the allowable increase and decrease for a non-basic variable is infinite and the other is the slack value.
- If a non-basic variable has zero slack value, then there may exist an alternative optimal solution.

Sample Problem 1

Let $A_1 \in R^{m\times n}, A_2 \in R^{m\times p}$ be two given matrices, and let $c_1 \in R^n, c_2 \in R^p$ be two given *non-negative vectors*. Consider the problem

$$\begin{array}{ll} \min & c_1^T x_1 + c_2^T x_2 \\ \text{s.t.} & A_1x_1 + A_2x_2 = b \\ & x_1, \quad x_2 \geq 0, \end{array}$$

and assume it is *feasible*.

(a) The problem has an optimal solution. Why?

(b) Let (x_1^*, x_2^*) be any optimal solution to the problem. Show that if a solution to the problem achieves an objective value $b^T y$ such that

$$A_1^T y \leq \alpha_1 c_1$$

$$A_2^T y \leq \alpha_2 c_2$$

where α_1 and α_2 are two scalars greater than or equal to 1, then

$$b^T y \leq \alpha_1 \cdot c_1^T x_1^* + \alpha_2 \cdot c_2^T x_2^*.$$

(α_1, α_2) is usually called the bi-factor approximation ratio and used in approximating algorithms for combinatorial optimization.

(a) Consider the dual problem:

$$\begin{array}{ll} \max & b^T y \\ \text{subject to} & A_1^T y \le c_1, \\ & A_2^T y \le c_2. \end{array}$$

Since $c_1 \ge 0$ and $c_2 \ge 0$, $y = 0$ is a feasible point for the dual. By LP duality, since the primal and dual problems are feasible, both must have optimal solutions.

(b) Let (x_1^*, x_2^*) be a primal optimal solution, and let (x_1, x_2) be a primal solution with value $c_1^T x_1 + c_2^T x_2 = b^T y$, where y satisfies

$$A_1^T y \leq \alpha_1 c_1,$$

$$A_2^T y \leq \alpha_2 c_2.$$

Since (x_1^*, x_2^*) is primal feasible, we must have $A_1 x_1^* + A_2 x_2^* = b$, and $x_1^*, x_2^* \geq 0$. Then,

$$(A_1^T y)^T x_1^* \leq \alpha_1 c_1^T x_1^*,$$
$$(A_2^T y)^T x_2^* \leq \alpha_2 c_2^T x_2^*.$$

Finally,

$$b^T y = (A_1 x_1^* + A_2 x_2^*)^T y = (A_1^T y)^T x_1^* + (A_2^T y)^T x_2^* \leq \alpha_1 c_1^T x_1^* + \alpha_2 c_2^T x_2^*.$$

Sample Problem 2: Betting on Permutation

Let A^k be a betting matrix where each entry is either 0 or 1. If it is a winning order, the auction organizer need to pay $A^k \bullet M_\sigma$ dollar for each share, where σ is the winning permutation and M_σ is the permutation matrix associated with σ.

The auction organizer's total payment to the winners:

$$\sum_{k=1}^{n}(A^k \bullet M_\sigma)x_k,$$

so that the worst-case profit maximization problem is

$$\begin{array}{lll} \max & \pi^T\mathbf{x} - \max_\sigma\{\sum_{k=1}^n (A^k \bullet M_\sigma)x_k\} & \\ \text{s.t.} & \mathbf{x} & \leq \mathbf{q}, \\ & \mathbf{x} & \geq \mathbf{0}, \end{array}$$

which is equivalent to

$$\begin{array}{lcl}
\max & \pi^T \mathbf{x} - s & \\
\text{s.t.} & \max_\sigma \left\{ \left(\sum_{k=1}^n x^k A^k \right) \bullet M_\sigma \right\} - s & \leq 0, \\
& \mathbf{x} & \leq \mathbf{q}, \\
& \mathbf{x} & \geq \mathbf{0}.
\end{array}$$

How many possible σs?

Given x_k, the inner maximization problem is to solve

$$\begin{array}{rl} \max_M & \left(\sum_{k=1}^n (x^k A^k\right) \bullet M \\ \text{s.t.} & M \text{is a permunation matrix,} \end{array}$$

or

$$\begin{array}{rrl} \max_M & \left(\sum_{k=1}^n x^k A^k\right) \bullet M & \\ \text{s.t.} & M\mathbf{e} & = \mathbf{e}, \\ & M^T\mathbf{e} & = \mathbf{e}, \\ & M & \geq \mathbf{0}. \end{array}$$

This is called the assignment problem, a special case of the transportation problem.

Every basic feasible solution of the linear program is a permutation matrix.

What's the dual problem? Replace the inner maximization objective by its dual.