

Conic Linear Programming

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Conic LP

$$\begin{aligned} (CLP) \quad & \text{minimize} && \mathbf{c} \bullet \mathbf{x} \\ & \text{subject to} && \mathbf{a}_i \bullet \mathbf{x} = b_i, i = 1, 2, \dots, m, \mathbf{x} \in C, \end{aligned}$$

where C is a convex cone.

Linear Programming (LP): $\mathbf{c}, \mathbf{a}_i, \mathbf{x} \in \mathcal{R}^n$ and $C = \mathcal{R}_+^n$

Second-Order Cone Programming (SOCP): $\mathbf{c}, \mathbf{a}_i, \mathbf{x} \in \mathcal{R}^n$ and $C = SOC$

Semidefinite Programming (SDP): $\mathbf{c}, \mathbf{a}_i, \mathbf{x} \in \mathcal{M}^n$ and $C = \mathcal{M}_+^n$

Note that cone C can be a product of many (different) convex cones.

LP, SOCP and SDP Examples

$$\begin{array}{ll} \text{minimize} & 2x_1 + x_2 + x_3 \\ \text{subject to} & x_1 + x_2 + x_3 = 1, \\ & (x_1; x_2; x_3) \succeq \mathbf{0}. \end{array}$$

$$\begin{array}{ll} \text{minimize} & 2x_1 + x_2 + x_3 \\ \text{subject to} & x_1 + x_2 + x_3 = 1, \\ & \sqrt{x_2^2 + x_3^2} \leq x_1. \end{array}$$

$$\begin{array}{ll} \text{minimize} & 2x_1 + x_2 + x_3 \\ \text{subject to} & x_1 + x_2 + x_3 = 1, \\ & \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} \succeq \mathbf{0}, \end{array}$$

where for SDP:

$$\mathbf{c} = \begin{pmatrix} 2 & .5 \\ .5 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{a}_1 = \begin{pmatrix} 1 & .5 \\ .5 & 1 \end{pmatrix}.$$

Or

$$\begin{aligned} \text{minimize} \quad & 2x_1 + x_2 + x_3 + 2x_4 + x_5 + x_6 \\ \text{subject to} \quad & x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 1, \\ & (x_1; x_2; x_3) \geq \mathbf{0}, (x_4; x_5; x_6) \in SOCP. \end{aligned}$$

Convex Optimization or Convex Programming

Convex Optimization: minimize a convex function over a convex constraint set/region.

An important fact for CO: any **local** minimizer is a **global** minimizer.

$$\begin{aligned} (CO) \quad & \text{minimize} && c_0(\mathbf{x}) \\ & \text{subject to} && c_i(\mathbf{x}) \leq b_i, i = 1, 2, \dots, m, \end{aligned}$$

where $c_i(\mathbf{x})$, $i = 0, 1, \dots, m$, are **convex functions** of \mathbf{x} .

Proof. Let $\hat{\mathbf{x}}$ be a local minimizer and \mathbf{x}^* be the global minimizer such that $c_0(\hat{\mathbf{x}}) > c_0(\mathbf{x}^*)$. Let $\mathbf{x}(\alpha) = \alpha\mathbf{x}^* + (1 - \alpha)\hat{\mathbf{x}}$. Then it is feasible and

$$c_0(\mathbf{x}(\alpha)) \leq \alpha c_0(\mathbf{x}^*) + (1 - \alpha)c_0(\hat{\mathbf{x}}) < c_0(\hat{\mathbf{x}}), \forall \alpha > 0.$$

This contradicts to $\hat{\mathbf{x}}$ being a local minimizer, since α can be small enough such that $\mathbf{x}(\alpha)$ is in the neighborhood of $\hat{\mathbf{x}}$.

Convex Optimization is equivalent to CLP

The convex program can be rewritten as

$$\begin{aligned}
 (CO) \quad & \text{minimize} && \alpha \\
 & \text{subject to} && c_0(\mathbf{x}) - \alpha \leq 0, \\
 & && c_i(\mathbf{x}) - b_i \leq 0, i = 1, 2, \dots, m.
 \end{aligned}$$

Thus, it is **sufficient** to consider convex optimization in a form

$$\begin{aligned}
 (CO) \quad & \text{minimize} && \mathbf{c}^T \mathbf{x} \\
 & \text{subject to} && c_i(\mathbf{x}) \leq 0, i = 1, 2, \dots, m,
 \end{aligned}$$

where $c_i(\mathbf{x})$, $i = 1, \dots, m$, are convex functions of \mathbf{x} .

Consider set

$$C_i = \{(t; \mathbf{x}) : t > 0, tc_i(\mathbf{x}/t) \leq 0.\}$$

It is a **convex cone** !

Convex Optimization is equivalent to CLP continued

Then, (CO) can be **equivalently** written as

$$\begin{aligned} &\text{minimize} && (0; \mathbf{c}) \bullet (t; \mathbf{x}) \\ &\text{subject to} && (1; \mathbf{0}) \bullet (t; \mathbf{x}) = 1, \\ &&& (t; \mathbf{x}) \in C_1 \cap, \dots, \cap C_m. \end{aligned}$$

This is a **Conic LP** !

We now develop **theories** for (CLP).

Dual of Conic LP

The **dual problem** to

$$\begin{aligned}
 (CLP) \quad & \text{minimize} \quad \mathbf{c} \bullet \mathbf{x} \\
 & \text{subject to} \quad \mathbf{a}_i \bullet \mathbf{x} = b_i, i = 1, 2, \dots, m, \mathbf{x} \in C.
 \end{aligned}$$

is

$$\begin{aligned}
 (CLD) \quad & \text{maximize} \quad \mathbf{b}^T \mathbf{y} \\
 & \text{subject to} \quad \sum_i^m y_i \mathbf{a}_i + \mathbf{s} = \mathbf{c}, \mathbf{s} \in C^*,
 \end{aligned}$$

where $y \in \mathcal{R}^m$ are the dual variables, \mathbf{s} is called the **dual slack** vector/matrix, and C^* is the dual cone of C .

Theorem 1 (*Weak duality theorem*)

$$\mathbf{c} \bullet \mathbf{x} - \mathbf{b}^T \mathbf{y} = \mathbf{x} \bullet \mathbf{s} \geq 0$$

for any **feasible** \mathbf{x} of (CLP) and (\mathbf{y}, \mathbf{s}) of (CLD).

Self-Dual Cones Again

Frequently, $C^* = C$, that is, they are **self-dual**.

The dual of the n -dimensional non-negative orthant, $\mathcal{R}_+^n = \{\mathbf{x} \in \mathcal{R}^n : \mathbf{x} \geq \mathbf{0}\}$, is \mathcal{R}_+^n ; it is **self-dual**.

The dual of the positive semi-definite matrices cone in \mathcal{M}^n , \mathcal{M}_+^n , is \mathcal{M}_+^n ; it is **self-dual**.

The dual of the second-order cone, $\{(t; \mathbf{x}) \in \mathcal{R}^{n+1} : t \geq \|\mathbf{x}\|\}$, is also the second-order cone; it is **self-dual**.

SOCP Examples

$$\begin{aligned} &\text{minimize} && \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \bullet \mathbf{x} \\ &\text{subject to} && \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \bullet \mathbf{x} = 1, \mathbf{x} \in \text{SOC}. \end{aligned}$$

Dual:

$$\begin{aligned} &\text{maximize} && y \\ &\text{subject to} && \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} - y \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \mathbf{s} \in \text{SOC}. \end{aligned}$$

SDP Examples

$$\text{minimize} \quad \begin{pmatrix} 2 & .5 \\ .5 & 1 \end{pmatrix} \bullet X$$

$$\text{subject to} \quad \begin{pmatrix} 1 & .5 \\ .5 & 1 \end{pmatrix} \bullet X = 1, \quad X \succeq \mathbf{0}.$$

Dual:

$$\begin{aligned} &\text{maximize} \quad y \\ &\text{subject to} \quad \begin{pmatrix} 2 & .5 \\ .5 & 1 \end{pmatrix} - y \cdot \begin{pmatrix} 1 & .5 \\ .5 & 1 \end{pmatrix} = S \succeq \mathbf{0}, \end{aligned}$$

Farkas' Lemma for General Cones?

Given \mathbf{a}_i , $i = 1, \dots, m$, and $\mathbf{b} \in \mathcal{R}^m$.

Then, the system $\{\mathbf{x} : \mathbf{a}_i \bullet \mathbf{x} = b_i, i = 1, \dots, m, \mathbf{x} \in C\}$ has a feasible solution \mathbf{x} if and only if that $-\sum_i^m y_i \mathbf{a}_i \in C^*$ and $\mathbf{b}^T \mathbf{y} > 0$ has no feasible solution \mathbf{y} ?

It is **necessary** but not **sufficient**!

Let's write equations in a **compact form**:

$$\mathcal{A}\mathbf{x} = (\mathbf{a}_1 \bullet \mathbf{x}; \dots; \mathbf{a}_m \bullet \mathbf{x}) \in \mathcal{R}^m$$

and

$$\mathcal{A}^T \mathbf{y} = \sum_i^m y_i \mathbf{a}_i.$$

Alternative Systems for General Cones?

Alternative System Pair I?:

$$\mathcal{A}\mathbf{x} = \mathbf{b}, \quad \mathbf{x} \in C,$$

and

$$-\mathcal{A}^T \mathbf{y} \in C^*, \quad \mathbf{b}^T \mathbf{y} = 1$$

Alternative System Pair II?:

$$\mathcal{A}\mathbf{x} = \mathbf{0}, \quad \mathbf{x} \in C, \quad \mathbf{c} \bullet \mathbf{x} = -1 (< 0)$$

and

$$\mathbf{c} - \mathcal{A}^T \mathbf{y} \in C^*$$

An SDP Cone Example when “Alternative System” failed

$$C = \mathcal{M}_+^2.$$

$$\mathbf{a}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{a}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and

$$\mathbf{b} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$

When Farkas' Lemma Holds for General Cones?

Let C be a closed convex cone in the rest of the course.

If there is \mathbf{y} such that $-\mathcal{A}^T \mathbf{y} \in \text{int } C^*$, then Alternative System Pair I is true:

$$\mathcal{A}\mathbf{x} = \mathbf{b}, \quad \mathbf{x} \in C,$$

and

$$-\mathcal{A}^T \mathbf{y} \in C^*, \quad \mathbf{b}^T \mathbf{y} = 1$$

And if there is \mathbf{x} such that $\mathcal{A}^T \mathbf{x} = \mathbf{0}$, $\mathbf{x} \in \text{int } C$, then Alternative System Pair II is true:

$$\mathcal{A}\mathbf{x} = \mathbf{0}, \quad \mathbf{x} \in C, \quad \mathbf{c} \bullet \mathbf{x} = -1 (< 0)$$

and

$$\mathbf{c} - \mathcal{A}^T \mathbf{y} \in C^*$$

Conic Linear Programming in Compact Form

$$\begin{aligned} (CLP) \quad & \text{minimize} && \mathbf{c} \bullet \mathbf{x} \\ & \text{subject to} && \mathcal{A}\mathbf{x} = \mathbf{b}, \\ & && \mathbf{x} \in C. \end{aligned}$$

$$\begin{aligned} (CLD) \quad & \text{maximize} && \mathbf{b}^T \mathbf{y} \\ & \text{subject to} && \mathcal{A}^T \mathbf{y} + \mathbf{s} = \mathbf{c}, \\ & && \mathbf{s} \in C^*. \end{aligned}$$

Denote by \mathcal{F}_p and \mathcal{F}_d the primal and dual **feasible sets**, respectively.

CLP Duality Theories

The weak duality theorem shows that a feasible solution to either problem yields a bound on the value of the other problem. We call $\mathbf{c} \bullet \mathbf{x} - \mathbf{b}^T \mathbf{y}$ the **duality gap**.

Corollary 1 Let $\mathbf{x}^* \in \mathcal{F}_p$ and $(\mathbf{y}^*, \mathbf{s}^*) \in \mathcal{F}_d$. Then, $\mathbf{c} \bullet \mathbf{x}^* = \mathbf{b}^T \mathbf{y}^*$ implies that \mathbf{x}^* is optimal for (CLP) and $(\mathbf{y}^*, \mathbf{s}^*)$ is optimal for (CLD).

Is the reverse also true? That is, given \mathbf{x}^* optimal for (CLP), then there is $(\mathbf{y}^*, \mathbf{s}^*)$ feasible for (CLD) and $\mathbf{c} \bullet \mathbf{x}^* = \mathbf{b}^T \mathbf{y}^*$?

This is called the **Strong Duality Theorem** and it is “true” for LP, but it is “False” in general cases.

SDP Example with a Duality Gap

$$\mathbf{c} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \mathbf{a}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \mathbf{a}_2 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

and

$$\mathbf{b} = \begin{pmatrix} 0 \\ 10 \end{pmatrix}.$$

When Strong Duality Theorems Holds for CLP

Theorem 2 (Strong duality theorem) Let \mathcal{F}_p and \mathcal{F}_d be non-empty and at least one of them has an interior. Then, \mathbf{x}^* is optimal for (CLP) and $(\mathbf{y}^*, \mathbf{s}^*)$ is optimal for (CLD) if and only if

$$\mathbf{c} \bullet \mathbf{x}^* = \mathbf{b}^T \mathbf{y}^*.$$

There are cases that the duality gap tends to zero but the optimal solution is not attainable.

Proof of Strong Duality Theorem for CLP

Let (CLP) have an interior feasible solution and a minimizer $\mathbf{x}^* \in \mathcal{F}_p$. Then,

$$\mathcal{A}\mathbf{x}' - \mathbf{b}\tau = \mathbf{0}, \quad (\mathbf{x}'; \tau) \in C \oplus \mathcal{R}_+, \quad \mathbf{c} \bullet \mathbf{x}' - (\mathbf{c}^T \mathbf{x}^*)\tau = -1 < 0$$

must have no **feasible** solution $(\mathbf{x}'; \tau)$. This is because otherwise, if $\tau > 0$, \mathbf{x}'/τ is **feasible** for (CLP) and $\mathbf{c} \bullet (\mathbf{x}'/\tau) < \mathbf{c} \bullet \mathbf{x}^*$, which is a **contradiction**; and if $\tau = 0$, $\mathbf{x}^* + \mathbf{x}'$ is **feasible** for (CLP) and $\mathbf{c} \bullet (\mathbf{x}^* + \mathbf{x}') = \mathbf{c} \bullet \mathbf{x}^* - 1 < \mathbf{c} \bullet \mathbf{x}^*$, which is also a **contradiction**.

Now, there is $(\mathbf{x}'; \tau) \in \text{int } C \oplus \mathcal{R}_+$ such that $\mathcal{A}\mathbf{x}' - \mathbf{b}\tau = \mathbf{0}$, since (CLP) has an interior feasible solution $(\mathbf{x} \in \text{int } C; \tau = 1)$. Thus, from the **CLP alternative system pair II**, there is \mathbf{y}^* feasible for

$$\mathbf{c} - \mathcal{A}^T \mathbf{y}^* \in C^*, \quad -\mathbf{c} \bullet \mathbf{x}^* + \mathbf{b}^T \mathbf{y}^* \geq 0.$$

Then, \mathbf{y}^* is feasible for (CLD) from the first inequality; and from the **weak duality theorem** we have $\mathbf{c} \bullet \mathbf{x}^* = \mathbf{b}^T \mathbf{y}^*$.

More Duality Theorem for CLP

Theorem 3 (CLP duality theorem) *If one of (CLP) or (CLD) is **unbounded** then the other has no feasible solution.*

*If (CLP) and (CLD) are both feasible, then both have bounded optimal objective values and the optimal objective values may have a **duality gap**.*

*If one of (CLP) or (CLD) has a strictly or **interior feasible** solution and it has an optimal solution, then the other is feasible and has an optimal solution with the same optimal value.*

Optimality Conditions for SDP

$$\begin{aligned}
 \mathbf{c} \bullet X - \mathbf{b}^T \mathbf{y} &= 0 \\
 \mathcal{A}X &= \mathbf{b} \\
 -\mathcal{A}^T \mathbf{y} - S &= -\mathbf{c} \quad , \\
 X, S &\succeq \mathbf{0}
 \end{aligned}
 \tag{1}$$

$$\begin{aligned}
 XS &= \mathbf{0} \\
 \mathcal{A}X &= \mathbf{b} \\
 -\mathcal{A}^T \mathbf{y} - S &= -\mathbf{c} \\
 X, S &\succeq \mathbf{0}
 \end{aligned}
 \tag{2}$$

Rank of SDP Solutions

At any optimal solution pair (X^*, S^*)

$$\text{rank}(X^*) + \text{rank}(S^*) \leq n.$$

If the equality holds, they are a **strictly complementary** solution pair.

There are optimal solutions of X^* and S^* such that the rank of X^* and the rank of S^* are **minimal**, respectively.

There are optimal solutions of X^* and S^* such that the rank of X^* and the rank of S^* are **maximal**, respectively.