

# **Interior Point Algorithms for Semidefinite Programming**

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## Semidefinite Programming (SDP)

$$\begin{aligned} (SDP) \quad & \text{Minimize} \quad C \bullet X \\ & \text{subject to} \quad \mathcal{A}X = \mathbf{b}, \quad X \succeq 0. \end{aligned}$$

The **dual** problem to (SDP) can be written as:

$$\begin{aligned} (SDD) \quad & \text{Maximize} \quad \mathbf{b}^T \mathbf{y} \\ & \text{subject to} \quad \mathcal{A}^T \mathbf{y} + S = C, \quad S \succeq 0. \end{aligned}$$

$$\mathcal{A}X = \begin{pmatrix} A_1 \bullet X \\ \dots \\ A_m \bullet X \end{pmatrix} \quad \text{and} \quad \mathcal{A}^T \mathbf{y} = \sum_{i=1}^m y_i A_i.$$

## An SDP Example and its LP Counter Part

$$\begin{aligned} (SDP) \quad & \text{minimize} && x_1 + x_2 + 2x_3 \\ & \text{subject to} && x_1 + x_2 + x_3 = 1, \\ & && \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} \succeq \mathbf{0}. \end{aligned}$$

$$\begin{aligned} (LP) \quad & \text{minimize} && x_1 + x_2 + 2x_3 \\ & \text{subject to} && x_1 + x_2 + x_3 = 1, \\ & && (x_1, x_2, x_3) \geq \mathbf{0}. \end{aligned}$$

## An SDP Example in Standard Form

$$\begin{aligned} (SDP) \quad & \text{minimize} && x_1 + x_2 + 2x_3 \\ & \text{subject to} && x_1 + x_2 + x_3 = 1, \\ & && \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} \succeq \mathbf{0}. \end{aligned}$$

$$C = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 2 \end{pmatrix}, A_1 = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}, \text{ and } b = 1.$$

## SDP Duality Theories

**Theorem 1** (*Weak duality theorem*) Let  $\mathcal{F}_p$  and  $\mathcal{F}_d$  be non-empty. Then,

$$C \bullet X \geq \mathbf{b}^T \mathbf{y} \quad \text{where } X \in \mathcal{F}_p, (\mathbf{y}, S) \in \mathcal{F}_d.$$

$$C \bullet X - \mathbf{b}^T \mathbf{y} = C \bullet X - (\mathcal{A}X)^T \mathbf{y} = C \bullet (C - \mathcal{A}^T \mathbf{y}) = X \bullet S \geq 0.$$

This theorem shows that a feasible solution to either problem yields a bound on the value of the other problem. We call  $C \bullet X - \mathbf{b}^T \mathbf{y}$  the **duality gap**, and  $X \bullet S$  the **complementarity gap**.

**SDP Example with a Duality Gap**

$$C = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

and

$$\mathbf{b} = \begin{pmatrix} 0 \\ 10 \end{pmatrix}.$$

## Strong Duality Theorem for SDP

**Theorem 2** (*Strong duality theorem*) Let  $\mathcal{F}_p$  and  $\mathcal{F}_d$  be non-empty and at least one of them has an interior. Then,  $X^*$  is optimal for (SDP) and  $(\mathbf{y}^*, S^*)$  is optimal for (SDD) if and only if

$$C \bullet X^* = \mathbf{b}^T \mathbf{y}^*.$$

**Theorem 3** (*SDP duality theorem*) If one of (SDP) or (SDD) is unbounded then the other has no feasible solution.

If (SDP) and (SDD) are both feasible, then both have bounded optimal objective values and the optimal objective values may have a duality gap.

If one of (SDP) or (SDD) has a strictly or interior feasible solution and it has an optimal solution, then the other is feasible and has an optimal solution with the same optimal value.

## Analytic Center of the SDP Set

$$\begin{aligned} (SDP) \quad & \text{Maximize} \quad \log \det(X) \\ & \text{subject to} \quad \mathcal{A}X = \mathbf{b}, X \succeq 0. \end{aligned}$$

$$\begin{aligned} (SDD) \quad & \text{Maximize} \quad \log \det(S) \\ & \text{subject to} \quad \mathcal{A}^T \mathbf{y} + S = C, S \succeq 0. \end{aligned}$$

## SDP with Barriers

Primal  $X(\mu)$ :

$$\begin{aligned} \text{minimize} \quad & C \bullet X - \mu \log \det(X) \\ \text{s.t.} \quad & \mathcal{A}X = \mathbf{b}, X \succ \mathbf{0}. \end{aligned}$$

Dual  $(\mathbf{y}(\mu), S(\mu))$ :

$$\begin{aligned} (D) \quad \text{maximize} \quad & \mathbf{b}^T \mathbf{y} + \mu \log \det(S) \\ \text{s.t.} \quad & \mathcal{A}^T \mathbf{y} + S = C, S \succ \mathbf{0}. \end{aligned}$$

## Central Path for SDP

The **central path** can be expressed as

$$\mathcal{C} = \left\{ (X, \mathbf{y}, S) \in \text{int } \mathcal{F} : XS = \frac{X \bullet S}{n} I \right\}.$$

$$XS = \mu I$$

$$\mathcal{A}X = \mathbf{b}$$

$$-\mathcal{A}^T \mathbf{y} - S = -\mathbf{c}$$

$$X, S \succ \mathbf{0}$$

## Primal-Dual Potential Functions for SDP

For any  $X \in \text{int } \mathcal{F}_p$  and  $(\mathbf{y}, S) \in \text{int } \mathcal{F}_d$ ,

$$\psi_{n+\rho}(X, S) := (n + \rho) \log(X \bullet S) - \log(\det(X) \cdot \det(S))$$

$$\psi_n(X, S) \geq n \log n.$$

$$\psi_{n+\rho}(X, S) = \rho \log(X \bullet S) + \psi_n(X, S) \geq \rho \log(X \bullet S) + n \log n.$$

Then, for  $\rho > 0$ ,  $\psi_{n+\rho}(X, S) \rightarrow -\infty$  implies that  $X \bullet S \rightarrow 0$ . More precisely, we have

$$X \bullet S \leq \exp\left(\frac{\psi_{n+\rho}(X, S) - n \log n}{\rho}\right).$$

## Important Lemmas

**Lemma 1** If  $\mathbf{d} \in \mathcal{R}^n$  and  $\|\mathbf{d}\|_\infty < 1$ , then

$$\mathbf{e}^T \mathbf{d} \geq \sum_{i=1}^n \log(1 + d_i) \geq \mathbf{e}^T \mathbf{d} - \frac{\|\mathbf{d}\|^2}{2(1 - \|\mathbf{d}\|_\infty)}.$$

**Lemma 2** If  $D \in \mathcal{M}^n$  and  $\|D\|_\infty < 1$ , then,

$$\text{Tr}(D) \geq \log \det(I + D) \geq \text{Tr}(D) - \frac{\|D\|^2}{2(1 - \|D\|_\infty)}.$$

## Primal-Dual (Symmetric) Algorithm for SDP

Once we have a pair  $(X, \mathbf{y}, S) \in \text{int } \mathcal{F}$  with  $\mu = S \bullet X/n$ , we can apply the **primal-dual Newton method** to generate a new iterate  $X^+$  and  $(\mathbf{y}^+, S^+)$  as follows: Solve for  $D_X$ ,  $\mathbf{d}_y$  and  $D_S$  from the system of linear equations:

$$\begin{aligned} D^{-1}D_X D^{-1} + D_S &= R := \gamma\mu X^{-1} - S, \\ \mathcal{A}D_X &= \mathbf{0}, \\ -\mathcal{A}^T \mathbf{d}_y - D_S &= \mathbf{0}, \end{aligned} \tag{1}$$

where

$$D = X^{.5} (X^{.5} S X^{.5})^{-.5} X^{.5}.$$

Note that  $D_S \bullet D_X = 0$ .

## Primal-Dual Scaling

$$\begin{aligned}
 D_{X'} + D_{S'} &= R', \\
 \mathcal{A}' D_{X'} &= \mathbf{0}, \\
 -\mathcal{A}'^T \mathbf{d}_y - D_{S'} &= \mathbf{0},
 \end{aligned} \tag{2}$$

where

$$D_{X'} = D^{-.5} D_X D^{-.5}, \quad D_{S'} = D^{.5} D_S D^{.5}, \quad R' = D^{.5} (\gamma \mu X^{-1} - S) D^{.5},$$

and

$$\mathcal{A}' = \begin{pmatrix} A'_1 \\ A'_2 \\ \dots \\ A'_m \end{pmatrix} := \begin{pmatrix} D^{.5} A_1 D^{.5} \\ D^{.5} A_2 D^{.5} \\ \dots \\ D^{.5} A_m D^{.5} \end{pmatrix}.$$

Again, we have  $D_{S'} \bullet D_{X'} = 0$ , and

$$\mathbf{d}_y = (\mathcal{A}' \mathcal{A}'^T)^{-1} \mathcal{A}' R', \quad D_{S'} = -\mathcal{A}'^T \mathbf{d}_y, \quad \text{and} \quad D_{X'} = R' - D_{S'}.$$

Or, we have

$$D_S = -\mathcal{A}^T \mathbf{d}_y \quad \text{and} \quad D_X = D(R - D_S)D.$$

## The Bound on Potential Reduction

$$V^{1/2} = D^{-.5} X D^{-.5} = D^{.5} S D^{.5} \in \text{int } \mathcal{M}_+^n.$$

Then, we can verify that  $S \bullet X = I \bullet V$ .

**Lemma 3** Let the direction  $D_X$ ,  $\mathbf{d}_y$  and  $D_S$  be generated by equation (1) with  $\gamma = n/(n + \rho)$ , and let

$$\theta = \frac{\alpha}{\|V^{-1/2}\|_\infty \left\| \frac{I \bullet V}{n+\rho} V^{-1/2} - V^{1/2} \right\|}, \quad (3)$$

where  $\alpha$  is a positive constant less than 1. Let

$$X^+ = X + \theta D_X, \quad \mathbf{y}^+ = \mathbf{y} + \theta \mathbf{d}_y, \quad \text{and} \quad S^+ = S + \theta D_S.$$

Then, we have  $(X^+, \mathbf{y}^+, S^+) \in \text{int } \mathcal{F}$  and

$$\begin{aligned} & \psi_{n+\rho}(X^+, S^+) - \psi_{n+\rho}(X, S) \\ & \leq -\alpha \frac{\|V^{-1/2} - \frac{n+\rho}{I \bullet V} V^{1/2}\|}{\|V^{-1/2}\|_\infty} + \frac{\alpha^2}{2(1-\alpha)}. \end{aligned}$$

## A Technical Lemma and the Convergence

**Lemma 4** Let  $V \in \text{int } \mathcal{M}_+^n$  and  $\rho \geq \sqrt{n}$ . Then,

$$\frac{\|V^{-1/2} - \frac{n+\rho}{I \bullet V} V^{1/2}\|}{\|V^{-1/2}\|_\infty} \geq \sqrt{3/4}.$$

From the two lemmas we have

$$\begin{aligned} & \psi_{n+\rho}(X^+, S^+) - \psi_{n+\rho}(X, S) \\ & \leq -\alpha \sqrt{3/4} + \frac{\alpha^2}{2(1-\alpha)} = -\delta \end{aligned}$$

for a constant  $\delta$ .

## Description of Algorithm

Given  $(X^0, \mathbf{y}^0, S^0) \in \text{int } \mathcal{F}$ . Set  $\rho \geq \sqrt{n}$  and  $k := 0$ .

**While**  $S^k \bullet X^k \geq \epsilon$  **do**

1. Set  $(X, S) = (X^k, S^k)$  and  $\gamma = n/(n + \rho)$  and compute  $(D_X, \mathbf{d}_y, D_S)$  from (1).
2. Let  $X^{k+1} = X^k + \bar{\alpha}D_X$ ,  $\mathbf{y}^{k+1} = \mathbf{y}^k + \bar{\alpha}\mathbf{d}_y$ , and  $S^{k+1} = S^k + \bar{\alpha}D_S$ , where

$$\bar{\alpha} = \arg \min_{\alpha \geq 0} \psi_{n+\rho}(X^k + \alpha D_X, S^k + \alpha D_S).$$

3. Let  $k := k + 1$  and return to Step 1.

## Complexity of the Algorithm

**Corollary 1** Let  $\rho = \sqrt{n}$ . Then, the Algorithm terminates in at most  $O(\sqrt{n} \log(C \bullet X^0 - \mathbf{b}^T \mathbf{y}^0) / \epsilon)$  iterations with

$$C \bullet X^k - \mathbf{b}^T \mathbf{y}^k \leq \epsilon.$$

## Dual Scaling Algorithm for SDP

An open question is how to exploit the **sparsity** structure by polynomial interior-point algorithms so that they can also solve large-scale problems in practice.

1. The computational cost of each iteration in the dual algorithm is **less** than the cost of the primal-dual iterations.
2. In most **combinatorial** applications, we need only a lower bound for the optimal objective value of (SDP).
3. For large scale problems,  $S$  tends to be very **sparse and structured** since it is the **linear combination** of  $C$  and the  $A_i$ 's. This sparsity allows considerable savings in both **memory and computation time**.

## Dual Algorithm

$$\psi_{n+\rho}(X, S) = (n + \rho) \ln(X \bullet S) - \ln \det X - \ln \det S.$$

Let  $\bar{z} = C \bullet X$  for some feasible  $X$  and consider the dual potential function

$$\phi(\mathbf{y}, \bar{z}) = (n + \rho) \ln(\bar{z} - \mathbf{b}^T \mathbf{y}) - \ln \det S.$$

Its gradient vector is

$$\nabla \phi(\mathbf{y}, \bar{z}) = -\frac{n + \rho}{\bar{z} - \mathbf{b}^T \mathbf{y}} \mathbf{b} + \mathcal{A}S^{-1}. \quad (4)$$

## Ellipsoid Constrained Problem

$$\begin{aligned} \text{Minimize} \quad & \nabla \phi^T(\mathbf{y}^k, \bar{z}^k)(\mathbf{y} - \mathbf{y}^k) \\ \text{Subject to} \quad & \| (S^k)^{-.5} (\mathcal{A}^T(\mathbf{y} - \mathbf{y}^k)) (S^k)^{-.5} \| \leq \alpha, \end{aligned} \tag{5}$$

where  $\alpha$  is a positive constant less than 1. For simplicity, in what follows we let

$$\Delta^k = \bar{z}^k - \mathbf{b}^T \mathbf{y}^k.$$

## Optimality Conditions

The optimality conditions state that the minimum point,  $\mathbf{y}^{k+1}$ , of this convex problem satisfies

$$M^k(\mathbf{y}^{k+1} - \mathbf{y}^k) + \beta \nabla \phi(\mathbf{y}^k, \bar{\mathbf{z}}^k) = 0 \quad (6)$$

for a positive value of  $\beta$ , where

$$M^k = \begin{pmatrix} A_1(S^k)^{-1} \bullet (S^k)^{-1} A_1 & \cdots & A_1(S^k)^{-1} \bullet (S^k)^{-1} A_m \\ \vdots & \ddots & \vdots \\ A_m(S^k)^{-1} \bullet (S^k)^{-1} A_1 & \cdots & A_m(S^k)^{-1} \bullet (S^k)^{-1} A_m \end{pmatrix}$$

The matrix  $M^k$  is a **Gram matrix** and is **positive definite** when  $S^k \succ 0$  and the  $A_i$ 's are linearly independent.

## Optimizer of the Problem

Using the **ellipsoidal** constraint, the minimal solution,  $\mathbf{y}^{k+1}$ , of (5) is given by

$$\mathbf{y}^{k+1} - \mathbf{y}^k = \frac{\alpha}{\sqrt{\nabla\phi^T(\mathbf{y}^k, \bar{\mathbf{z}}^k)(M^k)^{-1}\nabla\phi(\mathbf{y}^k, \bar{\mathbf{z}}^k)}} \mathbf{d}(\bar{\mathbf{z}}^k)_y \quad (7)$$

where

$$\mathbf{d}(\bar{\mathbf{z}}^k)_y = -(M^k)^{-1}\nabla\phi(\mathbf{y}^k, \bar{\mathbf{z}}^k). \quad (8)$$

## Generating $M$

Generally,  $M_{ij}^k = A_i(S^k)^{-1} \bullet (S^k)^{-1} A_j$ . When  $A_i = \mathbf{a}_i \mathbf{a}_i^T$ , the Gram matrix can be rewritten in the form

$$M^k = \begin{pmatrix} (\mathbf{a}_1^T (S^k)^{-1} \mathbf{a}_1)^2 & \cdots & (\mathbf{a}_1^T (S^k)^{-1} \mathbf{a}_m)^2 \\ \vdots & \ddots & \vdots \\ (\mathbf{a}_m^T (S^k)^{-1} \mathbf{a}_1)^2 & \cdots & (\mathbf{a}_m^T (S^k)^{-1} \mathbf{a}_m)^2 \end{pmatrix} \quad (9)$$

and

$$\mathcal{A}(S^k)^{-1} = \begin{pmatrix} \mathbf{a}_1^T (S^k)^{-1} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_m^T (S^k)^{-1} \mathbf{a}_m \end{pmatrix}.$$

This matrix can be computed very quickly without computing, or saving,  $(S^k)^{-1}$ .

## Potential Reduction

$$\nabla \phi^T(\mathbf{y}^k, \bar{z}^k) \mathbf{d}(\bar{z}^k)_y = -\|P(\bar{z}^k)\|^2 \quad (10)$$

$$\phi(\mathbf{y}^{k+1}, \bar{z}^k) - \phi(\mathbf{y}^k, \bar{z}^k) \leq -\alpha \|P(\bar{z}^k)\| + \frac{\alpha^2}{2(1-\alpha)}. \quad (11)$$

## Primal Update

To find a feasible **primal solution**  $X$ , we solve the least squares problem

$$\begin{aligned} \text{Minimize} \quad & \| (S^k)^{.5} X (S^k)^{.5} - \frac{\Delta^k}{(n+\rho)} I \|^2 \\ \text{Subject to} \quad & \mathcal{A}X = \mathbf{b}. \end{aligned} \tag{12}$$

The answer to (12) is a **by-product** of computing (8), given explicitly by

$$X(\bar{z}^k) = \frac{\Delta^k}{(n+\rho)} (S^k)^{-1} (\mathcal{A}^T \mathbf{d}(\bar{z}^k)_y + S^k) (S^k)^{-1}. \tag{13}$$

## Primal Objective Value

$$\begin{aligned}
 C \bullet X(\bar{z}^k) &= \mathbf{b}^T \mathbf{y}^k + X(\bar{z}^k) \bullet S^k \\
 &= \mathbf{b}^T \mathbf{y}^k + \text{Tr} \left( \frac{\Delta^k}{(n+\rho)} (S^k)^{-1} (\mathcal{A}^T \mathbf{d}(\bar{z}^k)_y + S^k) (S^k)^{-1} S^k \right) \\
 &= \mathbf{b}^T \mathbf{y}^k + \frac{\Delta^k}{(n+\rho)} \text{Tr} \left( (S^k)^{-1} \mathcal{A}^T \mathbf{d}(\bar{z}^k)_y + I \right) \\
 &= \mathbf{b}^T \mathbf{y}^k + \frac{\Delta^k}{(n+\rho)} \left( \mathbf{d}(\bar{z}^k)_y^T (\mathcal{A}(S^k)^{-1}) + n \right)
 \end{aligned}$$

Since the vectors  $\mathcal{A}(S^k)^{-1}$  and  $\mathbf{d}(\bar{z}^k)_y$  were previously found in calculating the **dual step** direction, the cost of computing a primal objective value is the cost of a **vector dot product**!

## Result for Primal

Defining

$$P(\bar{z}^k) = \frac{(n + \rho)}{\Delta^k} (S^k)^{.5} X(\bar{z}^k) (S^k)^{.5} - I, \quad (14)$$

we have the following lemma:

**Lemma 5** Let  $\mu^k = \frac{\Delta^k}{n} = \frac{\bar{z}^k - \mathbf{b}^T \mathbf{y}^k}{n}$ ,  $\mu = \frac{X(\bar{z}^k) \bullet S^k}{n} = \frac{C \bullet X(\bar{z}^k) - \mathbf{b}^T \mathbf{y}^k}{n}$ ,  
 $\rho \geq \sqrt{n}$ , and  $\alpha < 1$ . If

$$\|P(\bar{z}^k)\| < \min\left(\alpha \sqrt{\frac{n}{n + \alpha^2}}, 1 - \alpha\right), \quad (15)$$

then the following three inequalities hold:

1.  $X(\bar{z}^k) \succ 0$ ;
2.  $\|(S^k)^{.5} X(\bar{z}^k) (S^k)^{.5} - \mu I\| \leq \alpha \mu$ ;
3.  $\mu \leq (1 - .5\alpha/\sqrt{n})\mu^k$ .

**Theorem 4** *Either*

$$\psi_{n+\rho}(X^k, S^{k+1}) \leq \psi_{n+\rho}(X^k, S^k) - \delta$$

*or*

$$\psi_{n+\rho}(X^{k+1}, S^k) \leq \psi_{n+\rho}(X^k, S^k) - \delta,$$

*where*  $\delta > 1/20$ .

## Description of Algorithm

**DUAL ALGORITHM.** Given an upper bound  $\bar{z}^0$  and a dual point  $(\mathbf{y}^0, S^0)$  such that  $S^0 = C - \mathcal{A}^T \mathbf{y}^0 \succ 0$ , set  $k = 0$ ,  $\rho > \sqrt{n}$ ,  $\alpha \in (0, 1)$ , and do the following:

**while**  $\bar{z}^k - \mathbf{b}^T \mathbf{y}^k \geq \epsilon$  **do**

**begin**

1. Compute  $\mathcal{A}(S^k)^{-1}$  and the Gram matrix  $M^k$  (9) using Algorithm M or M'.
2. Solve (8) for the dual step direction  $\mathbf{d}(\bar{z}^k)_y$ .
3. Calculate  $\|P(\bar{z}^k)\|$  using (10).
4. **If** (15) is true, **then**  $X^{k+1} = X(\bar{z}^k)$ ,  $\bar{z}^{k+1} = C \bullet X^{k+1}$ , and  $(\mathbf{y}^{k+1}, S^{k+1}) = (\mathbf{y}^k, S^k)$ ;  
**else**  $\mathbf{y}^{k+1} = \mathbf{y}^k + \frac{\alpha}{\|P(\bar{z}^k)\|} \mathbf{d}(\bar{z}^{k+1})_y$ ,  $S^{k+1} = C - \mathcal{A}^T \mathbf{y}^{k+1}$ ,  
 $X^{k+1} = X^k$ , and  $\bar{z}^{k+1} = \bar{z}^k$ .

**endif**

5.  $k := k + 1.$

**end**

## Complexity of the Algorithm

**Corollary 2** Let  $\rho = \sqrt{n}$ . Then, the Algorithm terminates in at most  $O(\sqrt{n} \log(C \bullet X^0 - \mathbf{b}^T \mathbf{y}^0)/\epsilon)$  iterations with

$$C \bullet X^k - \mathbf{b}^T \mathbf{y}^k \leq \epsilon.$$

## Recall Homogeneous and Self-Dual Algorithm

- It solves the linear programming problem without any regularity assumption concerning the existence of **optimal, feasible, or interior feasible** solutions, while it retains the currently best complexity result
- It can start at any positive primal-dual pair, **feasible or infeasible**, near the central ray of the positive orthant (cone), and it does not use any big  $M$  penalty parameter or lower bound.
- Each iteration solves a system of linear equations whose dimension is almost the **same** as that solved in the standard (primal-dual) interior-point algorithms.
- If the LP problem has a solution, the algorithm generates a sequence that approaches **feasibility and optimality** simultaneously; if the problem is infeasible or unbounded, the algorithm will produce an **infeasibility certificate** for at least one of the primal and dual problems.

## A HSD linear program

Given any  $X^0 = I \succ \mathbf{0}$ ,  $S^0 = I \succ \mathbf{0}$ , and  $\mathbf{y}^0 = \mathbf{0}$ , we formulate

$$\begin{aligned}
 (HSDP) \quad & \min && (n+1)\theta \\
 & \text{s.t.} && \mathcal{A}X - \mathbf{b}\tau + \bar{\mathbf{b}}\theta = \mathbf{0}, \\
 & && -\mathcal{A}^T \mathbf{y} + C\tau - \bar{C}\theta \succeq \mathbf{0}, \\
 & && \mathbf{b}^T \mathbf{y} - C \bullet X + \bar{z}\theta \geq 0, \\
 & && -\bar{\mathbf{b}}^T \mathbf{y} + \bar{C} \bullet X - \bar{z}\tau = -(n+1), \\
 & && \mathbf{y} \text{ free, } X \succeq \mathbf{0}, \tau \geq 0, \theta \text{ free,}
 \end{aligned}$$

where

$$\bar{\mathbf{b}} = \mathbf{b} - \mathcal{A}I, \quad \bar{C} = C - I, \quad \bar{z} = C \bullet I + 1.$$

## Software Implementation

**SEDUMI:** <http://sedumi.mcmaster.ca/>

**DSDP:** <http://www-unix.mcs.anl.gov/DSDP/>

**MOSEK:** [http://www.mosek.com/products\\_mosek.html](http://www.mosek.com/products_mosek.html)

**CVX:** <http://www.stanford.edu/~boyd/cvx/>

**More SDP Codes:** <http://plato.asu.edu/ftp/sdplib.html>