

## **Interior Point Algorithms IV**

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## Termination with Exact Optimizers

- The first is a “cross-over” procedure to find a **basic feasible solution** (BFS, corner point) whose objective value is at least as good as the current interior point. Let  $A, \mathbf{b}, \mathbf{c}$  be integers and  $L$  be their **bit length**, and let a second best BFS solution be  $\mathbf{x}^{2nd}$  and the **optimal objective value** be  $z^*$ . Then

$$\mathbf{c}^T \mathbf{x}^{2nd} - z^* > 2^{-L}.$$

Thus, one can terminate interior-point algorithm when

$$\mathbf{c}^T \mathbf{x}^k - \mathbf{b}^T \mathbf{y}^k \leq 2^{-L}.$$

- The second approach is to compute a **strictly complementary solution pair**. The method uses the primal-dual interior-point pair to identify the **strict complementarity partition**  $(P^*, Z^*)$  and then “purify or project” the primal interior solution onto the **primal optimal face** and the dual interior solution onto

the **dual optimal face**, based on the following theorem:

**Theorem 1** *Given an interior solution  $\mathbf{x}^k$  and  $\mathbf{s}^k$  in the solution sequence generated by an interior-point algorithm, define*

$$P^k = \{j : x_j^k \geq s_j^k, \forall j\} \quad \text{and} \quad Z^k = \{1, \dots, n\} \setminus P^k.$$

*Then, we have  $P^k = P^*$  whenever*

$$\mathbf{c}^T \mathbf{x}^k - \mathbf{b}^T \mathbf{y}^k \leq 2^{-L}.$$

Thus, the **worst-case iteration bound** for interior-point algorithms is  $O(\sqrt{n}L)$  if the initial point pair  $(\mathbf{x}^0)^T \mathbf{s}^0 \leq 2^L$ .

## Initialization

- Combining the primal and dual into a single **linear feasibility** problem, then applying LP algorithms to find a feasible point of the problem. Theoretically, this approach can retain the currently best complexity result.
- The **big  $M$**  method, i.e., add one or more artificial column(s) and/or row(s) and a huge penalty parameter  $M$  to force solutions to become feasible during the algorithm.
- **Phase I-then-Phase II method**, i.e., first try to find a feasible point (and possibly one for the dual problem), and then start to look for an optimal solution if the problem is feasible and bounded.
- **Combined Phase I-Phase II method**, i.e., approach feasibility and optimality simultaneously. To our knowledge, the “best” complexity of this approach is  $O(n \log(R/\epsilon))$ .

## Homogeneous and Self-Dual Algorithm

- It solves the linear programming problem without any regularity assumption concerning the existence of **optimal, feasible, or interior feasible** solutions, while it retains the currently best complexity result
- It can start at any positive primal-dual pair, **feasible or infeasible**, near the central ray of the positive orthant (cone), and it does not use any big  $M$  penalty parameter or lower bound.
- Each iteration solves a system of linear equations whose dimension is almost the **same** as that solved in the standard (primal-dual) interior-point algorithms.
- If the LP problem has a solution, the algorithm generates a sequence that approaches **feasibility and optimality** simultaneously; if the problem is infeasible or unbounded, the algorithm will produce an **infeasibility certificate** for at least one of the primal and dual problems.

## Primal-Dual Alternative Systems

A pair of LP has **two alternatives**

$$\begin{array}{ll}
 \text{(Solvable)} & \mathbf{Ax} - \mathbf{b} = \mathbf{0} \\
 & -\mathbf{A}^T \mathbf{y} + \mathbf{c} \geq \mathbf{0}, \\
 & \mathbf{b}^T \mathbf{y} - \mathbf{c}^T \mathbf{x} = 0, \\
 & \mathbf{y} \text{ free, } \mathbf{x} \geq \mathbf{0}
 \end{array}
 \quad \text{or} \quad
 \begin{array}{ll}
 \text{(Infeasible)} & \mathbf{Ax} = \mathbf{0} \\
 & -\mathbf{A}^T \mathbf{y} \geq \mathbf{0}, \\
 & \mathbf{b}^T \mathbf{y} - \mathbf{c}^T \mathbf{x} > 0, \\
 & \mathbf{y} \text{ free, } \mathbf{x} \geq \mathbf{0}
 \end{array}$$

## An Integrated Homogeneous System

The two alternative systems can be **homogenized** as one:

$$\begin{aligned} (HP) \quad Ax - \mathbf{b}\tau &= \mathbf{0} \\ -A^T \mathbf{y} + \mathbf{c}\tau &= \mathbf{s} \geq \mathbf{0}, \\ \mathbf{b}^T \mathbf{y} - \mathbf{c}^T \mathbf{x} &= \kappa \geq 0, \\ \mathbf{y} \text{ free, } (\mathbf{x}; \tau) &\geq \mathbf{0} \end{aligned}$$

where the **two alternatives** are

$$(\text{Solvable}) : (\tau > 0, \kappa = 0) \quad \text{or} \quad (\text{Infeasible}) : (\tau = 0, \kappa > 0)$$

## The Homogeneous System is Self-Dual

$$\begin{array}{ll}
 (HP) & Ax - b\tau = \mathbf{0}, (y') \\
 & -A^T y + c\tau = \mathbf{s} \geq \mathbf{0}, (x') \\
 & b^T y - c^T x = \kappa \geq 0, (\tau') \\
 & y \text{ free}, (x; \tau) \geq \mathbf{0} \\
 (HD) & Ax' - b\tau' = \mathbf{0}, \\
 & A^T y' - c\tau' \leq \mathbf{0}, \\
 & -b^T y' + c^T x' \leq 0, \\
 & y' \text{ free}, (x'; \tau') \geq \mathbf{0}
 \end{array}$$

**Theorem 2** System (HP) is feasible (e.g. all zeros) and any feasible solution  $(y, x, \tau, s, \kappa)$  is *self-complementary*:

$$\mathbf{x}^T \mathbf{s} + \tau \kappa = 0.$$

Furthermore, it has a *strictly* self-complementary feasible solution

$$\begin{pmatrix} \mathbf{x} + \mathbf{s} \\ \tau + \kappa \end{pmatrix} > \mathbf{0},$$

## Let's Find Such a Feasible Solution

Given  $\mathbf{x}^0 = \mathbf{e} > \mathbf{0}$ ,  $\mathbf{s}^0 = \mathbf{e} > \mathbf{0}$ , and  $\mathbf{y}^0 = \mathbf{0}$ , we formulate

$$\begin{aligned}
 (HSDP) \quad & \min && \theta \\
 & \text{s.t.} && \\
 & & A\mathbf{x} & -\mathbf{b}\tau & +\bar{\mathbf{b}}\theta & = \mathbf{0}, \\
 & & -A^T\mathbf{y} & & +\mathbf{c}\tau & -\bar{\mathbf{c}}\theta & \geq \mathbf{0}, \\
 & & \mathbf{b}^T\mathbf{y} & -\mathbf{c}^T\mathbf{x} & & +\bar{z}\theta & \geq 0, \\
 & & \mathbf{y} \text{ free, } & \mathbf{x} \geq \mathbf{0}, & \tau \geq 0, & \theta \text{ free,}
 \end{aligned}$$

where

$$\bar{\mathbf{b}} = \mathbf{b} - A\mathbf{e}, \quad \bar{\mathbf{c}} = \mathbf{c} - \mathbf{e}, \quad \bar{z} = \mathbf{c}^T\mathbf{e} + 1.$$

But it may just give us the **all-zero solution**.

## A HSD linear program

Let's try to add one more constraint to **prevent the all-zero solution**

$$\begin{aligned}
 (\text{HSDP}) \quad & \min && (n+1)\theta \\
 & \text{s.t.} && Ax - \mathbf{b}\tau + \bar{\mathbf{b}}\theta = \mathbf{0}, \\
 & && -A^T \mathbf{y} + \mathbf{c}\tau - \bar{\mathbf{c}}\theta \geq \mathbf{0}, \\
 & && \mathbf{b}^T \mathbf{y} - \mathbf{c}^T \mathbf{x} + \bar{z}\theta \geq 0, \\
 & && -\bar{\mathbf{b}}^T \mathbf{y} + \bar{\mathbf{c}}^T \mathbf{x} - \bar{z}\tau = -(n+1), \\
 & && \mathbf{y} \text{ free, } \mathbf{x} \geq \mathbf{0}, \tau \geq 0, \theta \text{ free.}
 \end{aligned}$$

Note that the constraints of (HSDP) form a **skew-symmetric system** and the objective coefficient vector is the negative of the right-hand-side vector, so that it remains a **self-dual** linear program.

$(\mathbf{y} = \mathbf{0}, \mathbf{x} = \mathbf{e}, \tau = 1, \theta = 1)$  is a **strictly** feasible point for (HSDP).

$$\begin{aligned}
 (\text{HSDP}) \quad & \min && (n+1)\theta \\
 & \text{s.t.} && Ax - \mathbf{b}\tau + \bar{\mathbf{b}}\theta = \mathbf{0}, \\
 & && -A^T \mathbf{y} + \mathbf{c}\tau - \bar{\mathbf{c}}\theta = \mathbf{s} \geq \mathbf{0}, \\
 & && \mathbf{b}^T \mathbf{y} - \mathbf{c}^T \mathbf{x} + \bar{\mathbf{z}}\theta = \kappa \geq 0, \\
 & && -\bar{\mathbf{b}}^T \mathbf{y} + \bar{\mathbf{c}}^T \mathbf{x} - \bar{\mathbf{z}}\tau = -(n+1), \\
 & && \mathbf{y} \text{ free, } \mathbf{x} \geq \mathbf{0}, \tau \geq 0, \theta \text{ free.}
 \end{aligned}$$

Denote by  $\mathcal{F}_h$  the set of all points  $(\mathbf{y}, \mathbf{x}, \tau, \theta, \mathbf{s}, \kappa)$  that are feasible for (HSDP). Denote by  $\mathcal{F}_h^0$  the set of interior feasible points with  $(\mathbf{x}, \tau, \mathbf{s}, \kappa) > \mathbf{0}$  in  $\mathcal{F}_h$ . By combining the constraints, we can derive the last (equality) constraint as

$$\mathbf{e}^T \mathbf{x} + \mathbf{e}^T \mathbf{s} + \tau + \kappa - (n+1)\theta = (n+1),$$

which serves indeed as a **normalizing constraint** for (HSDP) to prevent the all-zero solution.

**Theorem 3** Consider problems (HSDP) and (HSDD).

i) (HSDD) has the same form as (HSDP), i.e., (HSDD) is simply (HSDP) with  $(\mathbf{y}, \mathbf{x}, \tau, \theta)$  being replaced by  $(\mathbf{y}', \mathbf{x}', \tau', \theta')$ .

ii) (HSDP) has a *strictly* feasible point

$$\mathbf{y} = \mathbf{0}, \quad x = \mathbf{e} > \mathbf{0}, \quad \tau = 1, \quad \theta = 1, \quad \mathbf{s} = \mathbf{e} > \mathbf{0}, \quad \kappa = 1.$$

iii) (HSDP) has an optimal solution and its optimal solution set is *bounded*.

iv) The optimal value of (HSDP) is zero, and

$$(\mathbf{y}, \mathbf{x}, \tau, \theta, \mathbf{s}, \kappa) \in \mathcal{F}_h \quad \text{implies that} \quad (n+1)\theta = \mathbf{x}^T \mathbf{s} + \tau \kappa.$$

v) There is an optimal solution  $(\mathbf{y}^*, \mathbf{x}^*, \tau^*, \theta^* = 0, \mathbf{s}^*, \kappa^*) \in \mathcal{F}_h$  such that

$$\begin{pmatrix} \mathbf{x}^* + \mathbf{s}^* \\ \tau^* + \kappa^* \end{pmatrix} > \mathbf{0},$$

which we call a *strictly self-complementary solution*. (Similarly, we sometimes call an optimal solution to (HSDP) a *self-complementary solution*; the strict inequalities above need not hold.)

**Theorem 4** Let  $(\mathbf{y}^*, \mathbf{x}^*, \tau^*, \theta^* = 0, \mathbf{s}^*, \kappa^*)$  be a strictly self complementary solution for (HSDP).

- i) (LP) has a solution (*feasible and bounded*) if and only if  $\tau^* > 0$ . In this case,  $\mathbf{x}^* / \tau^*$  is an optimal solution for (LP) and  $(\mathbf{y}^* / \tau^*, \mathbf{s}^* / \tau^*)$  is an optimal solution for (LD).
- ii) (LP) has *no solution* if and only if  $\kappa^* > 0$ . In this case,  $\mathbf{x}^* / \kappa^*$  or  $\mathbf{s}^* / \kappa^*$  or both are certificates for proving *infeasibility*: if  $\mathbf{c}^T \mathbf{x}^* < 0$  then (LD) is infeasible; if  $-\mathbf{b}^T \mathbf{y}^* < 0$  then (LP) is infeasible; and if both  $\mathbf{c}^T \mathbf{x}^* < 0$  and  $-\mathbf{b}^T \mathbf{y}^* < 0$  then both (LP) and (LD) are infeasible.

**Theorem 5** i) For any  $\mu > 0$ , there is a unique  $(\mathbf{y}, \mathbf{x}, \tau, \theta, \mathbf{s}, \kappa)$  in  $\mathcal{F}_h^0$ , such that

$$\begin{pmatrix} X\mathbf{s} \\ \tau\kappa \end{pmatrix} = \mu\mathbf{e}.$$

ii) Let  $(\mathbf{d}_y, \mathbf{d}_x, d_\tau, d_\theta, \mathbf{d}_s, d_\kappa)$  be in the null space of the constraint matrix of (HSDP) after adding surplus variables  $\mathbf{s}$  and  $\kappa$ , i.e.,

$$\begin{aligned} Ad_x - \mathbf{b}d_\tau + \bar{\mathbf{b}}d_\theta &= \mathbf{0}, \\ -A^T \mathbf{d}_y + \mathbf{c}d_\tau - \bar{\mathbf{c}}d_\theta - \mathbf{d}_s &= \mathbf{0}, \\ \mathbf{b}^T \mathbf{d}_y - \mathbf{c}^T \mathbf{d}_x + \bar{\mathbf{z}}d_\theta - d_\kappa &= 0, \\ -\bar{\mathbf{b}}^T \mathbf{d}_y + \bar{\mathbf{c}}^T \mathbf{d}_x - \bar{\mathbf{z}}d_\tau &= 0. \end{aligned} \tag{1}$$

$$(\mathbf{d}_x)^T \mathbf{d}_s + d_\tau d_\kappa = 0.$$

## Endogenous Potential Function and Central Path

$$\psi_{n+\rho}(\mathbf{x}, \mathbf{s}, \tau, \kappa) := (n+1+\rho) \log(\mathbf{x}^T \mathbf{s} + \tau \kappa) - \sum_{j=1}^n \log(x_j s_j) - \log(\tau \kappa),$$

and

$$\mathcal{C} = \left\{ (\mathbf{y}, \mathbf{x}, \tau, \theta, \mathbf{s}, \kappa) \in \mathcal{F}_h^0 : \begin{pmatrix} X\mathbf{s} \\ \tau\kappa \end{pmatrix} = \frac{\mathbf{x}^T \mathbf{s} + \tau\kappa}{n+1} \mathbf{e} \right\}.$$

Obviously, the initial interior feasible point proposed in Theorem 3 is on the path with  $\mu = 1$  or  $(\mathbf{x}^0)^T \mathbf{s}^0 + \tau^0 \kappa^0 = n + 1$ .

## Solving (HSDP)

Consider solving the following **system of linear equations** for  $(\mathbf{d}_y, \mathbf{d}_x, d_\tau, d_\theta, \mathbf{d}_s, d_\kappa)$  that satisfies (1) and

$$\begin{pmatrix} X\mathbf{d}_s + S\mathbf{d}_x \\ \tau^k d_\kappa + \kappa^k d_\tau \end{pmatrix} = \gamma\mu\mathbf{e} - \begin{pmatrix} X\mathbf{s} \\ \tau\kappa \end{pmatrix}.$$

**Theorem 6** *The  $O(\sqrt{n} \log((\mathbf{x}^0)^T \mathbf{s}^0 / \epsilon))$  interior-point algorithm, coupled with a termination technique described above, generates a **strictly self-complementary solution** for (HSDP) in  $O(\sqrt{n}(\log(c(A, \mathbf{b}, \mathbf{c})) + \log n))$  iterations and  $O(n^3(\log(c(A, \mathbf{b}, \mathbf{c})) + \log n))$  operations, where  $c(A, \mathbf{b}, \mathbf{c})$  is a positive number depending on the data  $(A, \mathbf{b}, \mathbf{c})$ . If (LP) and (LD) have integer data with **bit length  $L$** , then by the construction, the data of (HSDP) remains integral and its length is  $O(L)$ . Moreover,  $c(A, \mathbf{b}, \mathbf{c}) \leq 2^L$ . Thus, the algorithm terminates in  $O(\sqrt{n}L)$  iterations and  $O(n^3L)$  operations.*

## Example

Consider the example where

$$A = \begin{pmatrix} -1 & 0 & 0 \end{pmatrix}, \quad b = 1, \quad \text{and} \quad \mathbf{c} = \begin{pmatrix} 0 & 1 & -1 \end{pmatrix}.$$

Then,

$$y^* = 2, \quad \mathbf{x}^* = (0, 2, 1)^T, \quad \tau^* = 0, \quad \theta^* = 0, \quad \mathbf{s}^* = (2, 0, 0)^T, \quad \kappa^* = 1$$

could be a strictly self-complementary solution generated for (HSDP) with

$$\mathbf{c}^T \mathbf{x}^* = 1 > 0, \quad by^* = 2 > 0.$$

Thus  $(y^*, \mathbf{s}^*)$  demonstrates the infeasibility of (LP), but  $\mathbf{x}^*$  doesn't show the infeasibility of (LD). Of course, if the algorithm generates instead  $\mathbf{x}^* = (0, 1, 2)^T$ , then we get demonstrated infeasibility of both.

## Software Implementation

Cplex

**SEDUMI**: <http://sedumi.mcmaster.ca/>

**MOSEK**: [http://www.mosek.com/products\\_mosek.html](http://www.mosek.com/products_mosek.html)

**IPOPT**: <https://projects.coin-or.org/Ipopt>

**sphs1bf**: Sparse Linear Programming Solver (Matlab .m file).