Applications of Analytic Center and IPA

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Combinatorial Auction: an order

The jth order is given as $(\mathbf{a}_j \in R^m_+, \ \pi_j \in R_+, \ q_j \in R_+)$: $\mathbf{a}_j$ is the betting indication vector where each entry is either 1 or 0

$$\mathbf{a}_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \dots \\ a_{mj} \end{pmatrix},$$

where 1 is winning state and 0 is non-winning state; π_j is the price limit for one unit of a contract, and q_j is the maximum number of units or shares the better like to own.

Combinatorial Auction: LP formulation

$$\begin{array}{lrl} \max & \pi^T \mathbf{x} - z & \\ \text{s.t.} & A\mathbf{x} - \mathbf{e} \cdot z & \leq \mathbf{0}, \\ & \mathbf{x} & \leq \mathbf{q}, \\ & \mathbf{x} & \geq \mathbf{0}. \end{array}$$

$$\begin{array}{lrl} \min & \mathbf{q}^T \mathbf{y} & \\ \text{s.t.} & A^T \mathbf{p} + \mathbf{y} & \geq \pi, \\ & \mathbf{e}^T \mathbf{p} & = 1, \\ & (\mathbf{p}, \mathbf{y}) & \geq \mathbf{0}. \end{array}$$

$\mathbf{p}$ represents the state price. But $\mathbf{p}$ may not be unique.

Combinatorial Auction: utility formulation

$$\begin{array}{lll} \max & \pi^T \mathbf{x} - z + u(\mathbf{s}) & \\ \text{s.t.} & A\mathbf{x} - \mathbf{e} \cdot z + \mathbf{s} & = \mathbf{0}, \\ & \mathbf{x} & \leq \mathbf{q}, \\ & (\mathbf{x}, \mathbf{s}) & \geq \mathbf{0}. \end{array}$$

where $u(\cdot)$ is a concave vaule function for the surplus profit $\mathbf{s} = \mathbf{e} \cdot z - A\mathbf{x}$ in each state i.

For example, $u(\mathbf{s}) = \sum_i \lambda_i s_i = \sum_i \lambda_i (z - \mathbf{a}_i \mathbf{x})$ where λ_i is organizer's probability distribution of winning states and $\mathbf{a}_i$ is the ith row vector of A.

Combinatorial Auction: barrier formulation

$$
\begin{array}{llll}
\max & \pi^T \mathbf{x} - z + \sum_i \theta_i \log(s_i) & & \\
\text{s.t.} & A\mathbf{x} - \mathbf{e} \cdot z + \mathbf{s} & = \mathbf{0}, \\
& \mathbf{x} & \leq \mathbf{q}, \\
& (\mathbf{x}, \mathbf{s}) & \geq \mathbf{0}.
\end{array}
$$

where θ_i represents initial seed money for state i.

For any given $\theta_i > 0$ for all i, the price vector is now unique.

Combinatorial Auction: Seed money control

One can set $\theta_i = \mu\theta_i^0 > 0$ where $\mu > 0$ may vary.

Then price vector is a function of μ written as $\mathbf{p}(\mu)$. As $\mu \rightarrow 0$, $\mathbf{p}(\mu) \rightarrow \mathbf{p}*$ where $\mathbf{p}*$ is vector in the interior of the optimal face of the dual of the original auction LP model.

Uniqueness of the optimizer

Consider more general problem:

$$\begin{array}{lll} \max & \mathbf{c}^T\mathbf{x} + \sum_j w_j \log x_j & \\ \text{s.t.} & A\mathbf{x} & = \mathbf{b}, \\ & \mathbf{x} & \geq \mathbf{0}; \end{array}$$

$$\begin{aligned} x_j p_j &= w_j, \forall j = 1, \ldots, n \\ A\mathbf{x} &= \mathbf{b} \\ A^T\mathbf{y} - \mathbf{p} &= \mathbf{c} \\ \mathbf{x}, \mathbf{p} &\geq \mathbf{0}. \end{aligned}$$

Theorem 1 *x_j and p_j are unique among all KKT solutions if $w_j > 0$.*

Proof

Let $\mathbf{x}^1, \mathbf{y}^1, \mathbf{p}^1$ and $\mathbf{x}^2, \mathbf{y}^2, \mathbf{p}^2$ be two KKT solutions where $x_1^1 > x_1^2 (> 0)$ for a given $w_1 > 0$. Then, from

$$x_1^1 p_1^1 = w_1 = x_1^2 p_1^2,$$

we must have $(0 <) p_1^1 < p_1^2$ so that $(x_1^1 - x_1^2) \cdot (p_1^1 - p_1^2) < 0$.

Since $(\mathbf{x}^1 - \mathbf{x}^2)^T (\mathbf{p}^1 - \mathbf{p}^2) = 0$ (?), there must be a $1 < i \leq n$ such that $(x_i^1 - x_i^2) \cdot (p_i^1 - p_i^2) > 0$. Then, we must have either $x_i^1 > x_i^2 \geq 0$ and $p_i^1 > p_i^2 \geq 0$, or $0 \leq x_i^1 < x_i^2$ and $0 \leq p_i^1 < p_i^2$. Either case contradicts

$$x_i^1 p_i^1 = w_i = x_i^2 p_i^2.$$

Uniqueness of the optimizer

Consider more general problem:

$$\begin{array}{lll} \max & \mathbf{c}^T\mathbf{x} + \frac{1}{2}\mathbf{x}^TQ\mathbf{x} + \sum_j w_j \log x_j & \\ \text{s.t.} & A\mathbf{x} & = \mathbf{b}, \\ & \mathbf{x} & \geq \mathbf{0}; \end{array}$$

$$\begin{array}{rcl} x_j p_j & = & w_j, \forall j = 1, \ldots, n \\ A\mathbf{x} & = & \mathbf{b} \\ -Q\mathbf{x} + A^T\mathbf{y} - \mathbf{p} & = & \mathbf{c} \\ \mathbf{x}, \mathbf{p} & \geq & \mathbf{0}; \end{array}$$

where Q is negative semidefinite on the null space of A.

Theorem 2 *x_j and p_j are unique among all KKT solutions if $w_j > 0$.*

Fisher's equilibrium price

Player $i \in B$'s optimization problem for given prices p_j, $j \in G$.

$$\begin{array}{ll} \text{maximize} & \mathbf{u}_i^T \mathbf{x}_i := \sum_{j \in G} u_{ij} x_{ij} \\ \text{subject to} & \mathbf{p}^T \mathbf{x}_i := \sum_{j \in G} p_j x_{ij} \le w_i, \\ & x_{ij} \ge 0, \quad \forall j, \end{array}$$

Assume that the amount of each good on the market is s_j.

The equilinitum price vector $\mathbf{p}$ is the one that for all $j \in G$, there is an optimal $\mathbf{x}(\mathbf{p})_i$ for every buyer i such that

$$\sum_{i \in B} x(\mathbf{p})_{ij} = s_j.$$

Example of Fisher's equilibrium price

Buyer $1, 2$'s optimization problems for given prices p_x, p_y.

$$\begin{array}{ll} \text{maximize} & 2x_1 + y_1 \\ \text{subject to} & p_x \cdot x_1 + p_y \cdot y_1 \leq 5, \\ & x_1, y_1 \geq 0; \\ \text{maximize} & 3x_2 + y_2 \\ \text{subject to} & p_x \cdot x_2 + p_y \cdot y_2 \leq 8, \\ & x_2, y_2 \geq 0. \end{array}$$

$$p_x = \frac{26}{3}, \quad p_y = \frac{13}{3}$$

$$x_1 = \frac{1}{13}, \ y_1 = 1, \ x_2 = \frac{12}{13}, \ y_2 = 0$$

Equilibrium price conditions

Player $i \in B$'s dual problem for given prices p_j, $j \in G$.

$$\begin{array}{ll} \text{minimize} & w_i y_i \\ \text{subject to} & \mathbf{p} y_i \geq \mathbf{u}_i,\; y_i \geq 0 \end{array}$$

The necessary and sufficient conditions for an equilibrium point $\mathbf{x}_i, \mathbf{p}$ are:

$$\begin{array}{rl} \mathbf{p}^T \mathbf{x}_i \leq w_i,\; \mathbf{x}_i \geq \mathbf{0}, & \forall i, \\ p_j y_i \geq u_{ij},\; y_i \geq 0, & \forall i, j, \\ \mathbf{u}_i^T \mathbf{x}_i = w_i y_i, & \forall i, \\ \sum_i x_{ij} \leq s_j, & \forall j. \end{array}$$

Equilibrium price conditions continued

These conditions can be represented by

$$\begin{array}{ll} \sum_j s_j p_j \le \sum_i w_i, \ \mathbf{x}_i \ge \mathbf{0}, & \forall i, \\ \frac{\mathbf{u}_i^T \mathbf{x}_i}{w_i} \cdot p_j \ge u_{ij}, & \forall i, j, \\ \sum_i x_{ij} \le s_j, & \forall j. \end{array}$$

since from the second inequality (after multiplying $x_i j$ to both sides and take sum over j) we have

$$\mathbf{p}^T \mathbf{x}_i \ge w_i, \ \forall i.$$

Then, from the rest conditions

$$\sum_i w_i \ge \sum_j s_j p_j \ge \sum_i \mathbf{p}^T \mathbf{x}_i \ge \sum_i w_i.$$

Thus, these conditions imply $\mathbf{p}^T \mathbf{x}_i = w_i, \ \forall i$.

Aggregate Social Optimization

$$
\begin{array}{ll}
\text{maximize} & \sum_{i\in B} w_i \log(\mathbf{u}_i^T \mathbf{x}_i) \\
\text{subject to} & \sum_{i\in B} x_{ij} \le s_j, \quad \forall j \in G \\
& x_{ij} \ge 0, \quad \forall i, j,
\end{array}
$$

Theorem 3 *(Eisenberg and Gale 1959) Optimal dual (Lagrange) multiplier vector of equality constraints is an equilibrium price vector.*

Optimality Conditions of the aggregated problem

$$
\begin{aligned}
w_i \frac{u_{ij}}{\mathbf{u}_i^T \mathbf{x}_i} &\le p_j, \quad \forall i, j \\
w_i \frac{u_{ij} x_{ij}}{\mathbf{u}_i^T \mathbf{x}_i} &= p_j x_{ij}, \quad \forall i, j \\
\sum_i x_{ij} &\le s_j, \quad \forall j \\
p_j \sum_i x_{ij} &\le p_j s_j, \quad \forall j \\
\mathbf{x}_i,\ \mathbf{p} &\ge \mathbf{0}.
\end{aligned}
$$

Let $y_i = \frac{\mathbf{u}_i^T \mathbf{x}_i}{w_i}$. Then, these conditions are identical to the equilibrium price conditions, since

$$
y_i = \frac{\mathbf{u}_i^T \mathbf{x}_i}{w_i} \ge \frac{u_{ij}}{p_j}, \ \forall i, j.
$$

Rewrite aggregate social optimization

$$
\begin{array}{ll}
\text{maximize} & \sum_{i \in B} w_i \log u_i \\
\text{subject to} & \sum_{j \in G} u_{ij}^T x_{ij} - u_i = 0, \quad \forall i \in B \\
& \sum_{i \in B} x_{ij} \le s_j, \quad \forall j \in G \\
& x_{ij} \ge 0,\ s_i \ge 0, \quad \forall i, j,
\end{array}
$$

This is called the weighted analytic center problem.

Question: Is the price vector $\mathbf{p}$ unique when at least one $u_{ij} > 0$ among $i \in B$ and $u_{ij} > 0$ among $j \in G$.

Aggregate Example

$$\begin{array}{ll} \text{maximize} & 5\log(2x_1+y_1)+8\log(3x_2+y_2) \\ \text{subject to} & x_1+x_2=1, \\ & y_1+y_2=1, \\ & x_1,x_2,y_1,y_2\geq 0. \end{array}$$

Or

$$\begin{array}{ll} \text{maximize} & 5\log(u_1)+8\log(u_2) \\ \text{subject to} & 2x_1+y_1-u_1=0, \\ & 3x_2+y_2-u_2=0, \\ & x_1+x_2=1, \\ & y_1+y_2=1, \\ & x_1,x_2,y_1,y_2\geq 0. \end{array}$$

Fisher's Model with Leontief's Utility

Player $i \in B$'s optimization problem for given prices p_j, $j \in G$.

$$\begin{array}{ll} \text{maximize} & \mathbf{u}_i(\mathbf{x}_i) := \min_{j \in G} \left\{ \frac{x_{ij}}{a_{ij}} \right\} \\ \text{subject to} & \mathbf{p}^T \mathbf{x}_i := \sum_{j \in G} p_j x_{ij} \le w_i, \\ & x_{ij} \ge 0, \quad \forall j, \end{array}$$

where a_{ij} is the given utility factor of player i for good j and x_{ij} represents the amount of good j bought by player i.

$$\frac{*}{0} := \infty.$$

Buy goods proportionally: $x_{ij} = a_{ij} u_i$, where u_i is the utility value of player i.

Example of Fisher's model with Leontief's Utility

Buyer $1, 2$'s optimization problems for given prices p_x, p_y.

Buyer 1:	maximize	$\min\{x_1/2,\ y_1\}$	
	subject to	$p_x \cdot x_1 + p_y \cdot y_1 \leq 5;$	and
Buyer 2:	maximize	$\min\{x_2,\ y_2/3\}$	
	subject to	$p_x \cdot x_2 + p_y \cdot y_2 \leq 8.$	

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}.$$

$p_x = 0, \quad p_y = 13$ and $x_1 \geq \frac{30}{39},\ y_1 = \frac{5}{13},\ x_2 = 1 - x_1 \geq \frac{8}{39},\ y_2 = \frac{8}{13}$

Reformulation

Player $i \in B$'s optimization problem for given prices p_j, $j \in G$.

$$\begin{array}{ll} \text{maximize} & u_i \\ \text{subject to} & \mathbf{p}^T\mathbf{x}_i := \sum_{j \in G} p_j x_{ij} \le w_i, \\ & a_{ij}u_i - x_{ij} \le 0, \quad \forall j. \end{array}$$

or simply

$$\begin{array}{ll} \text{maximize} & u_i \\ \text{subject to} & \left(\sum_{j \in G} p_j a_{ij}\right) u_i \le w_i. \end{array}$$

Optimality Conditions of the Fisher-Leontief Model

$$
\begin{aligned}
u_i \sum_j a_{ij} p_j &= w_i, \ \forall i \\
p_j (1 - \sum_i a_{ij} u_i) &= 0, \ \forall j \\
\sum_i a_{ij} u_i &\leq 1, \ \forall j \\
u_i, \ p_j &\geq 0, \ \forall i, j.
\end{aligned}
$$

There may be no rational equilibrium price.

Aggregate Social Optimization of Fisher's Model

The aggregate social optimization of Fisher's model becomes

$$\begin{array}{ll} \text{maximize} & \sum_i w_i \log u_i \\ \text{subject to} & A^T\mathbf{u} \le \mathbf{s}, \\ & \mathbf{u} \ge \mathbf{0}; \end{array}$$

where w_i is the budget of consumer i, u_i represents the utility value of consumer i, and A is the Leontief matrix.

The optimal Lagrange vector $\mathbf{p}^*$ of the constraints is the Fisher price vector.

If $(A^T\mathbf{u})_j < s_j$ for some j, then $p^*_j = 0$ or the jth good is "free" so that we can arbitrarily distribute any remaining supply among buyers to clear the market.

Fisher's Model with Productions

Let producer $k \in P$'s optimization problem for given prices p_j, $j \in G$. be

$$\begin{array}{ll} \text{maximize} & \mathbf{p}^T \mathbf{s}_k \\ \text{subject to} & B_k \mathbf{s}_k \leq \mathbf{b}_k, \\ & \mathbf{s}_k \geq \mathbf{0}, \end{array}$$

and its dual problem:

$$\begin{array}{ll} \text{minimize} & \mathbf{b}_k^T \mathbf{y}_k \\ \text{subject to} & B_k^T \mathbf{y}_k \geq \mathbf{p}, \ \mathbf{y}_k \geq \mathbf{0}. \end{array}$$

Equilibrium price conditions

The equilinitum price vector $\mathbf{p}$ is the one that for all $j \in G$, there is an optimal $\mathbf{x}(\mathbf{p})_i$ for every buyer i and $\mathbf{s}(\mathbf{p})_k$ for every producer k such that

$$\sum_{i \in B} x(\mathbf{p})_{ij} = \sum_{k \in P} \mathbf{s}(\mathbf{p})_{kj}.$$

The necessary and sufficient conditions for an equilibrium point $\mathbf{x}_i \geq \mathbf{0}, \mathbf{p} \geq \mathbf{0}, \mathbf{s}_k \geq \mathbf{0}$ are:

$$\begin{array}{rcl}
\sum_k \mathbf{b}_k^T \mathbf{y}_k \leq \sum_i w_i, & & \\
\frac{\mathbf{u}_i^T \mathbf{x}_i}{w_i} \cdot \mathbf{p} \geq \mathbf{u}_i, & & \forall i, \\
\sum_i \mathbf{x}_i \leq \sum_k \mathbf{s}_k, & & \\
B_k^T \mathbf{y}_k \geq \mathbf{p}, \ \mathbf{y}_k \geq \mathbf{0} & & \forall k, \\
B_k \mathbf{s}_k \leq \mathbf{b}_k, & & \forall k.
\end{array}$$

Note that these conditions imply $\mathbf{p}^T \mathbf{x}_i = w_i, \ \forall i$ and $\mathbf{p}^T \mathbf{s}_k = \mathbf{b}_k^T \mathbf{y}_k, \ \forall k$.

Aggregate Social Optimization

$$
\begin{array}{lll}
\text{maximize} & \sum_{i \in B} w_i \log(\mathbf{u}_i^T \mathbf{x}_i) & \\
\text{subject to} & \sum_{i \in B} \mathbf{x}_i - \sum_k \mathbf{s}_k & \leq \mathbf{0}, \\
& B_k \mathbf{s}_k & \leq \mathbf{b}_k, \ \forall k, \\
& \mathbf{x}_i, \mathbf{s}_k & \geq \mathbf{0}, \ \forall i, k.
\end{array}
$$

Theorem 4 *Optimal dual (Lagrange) multiplier vector of equality constraints is an equilibrium price vector?*