

The Simplex Method

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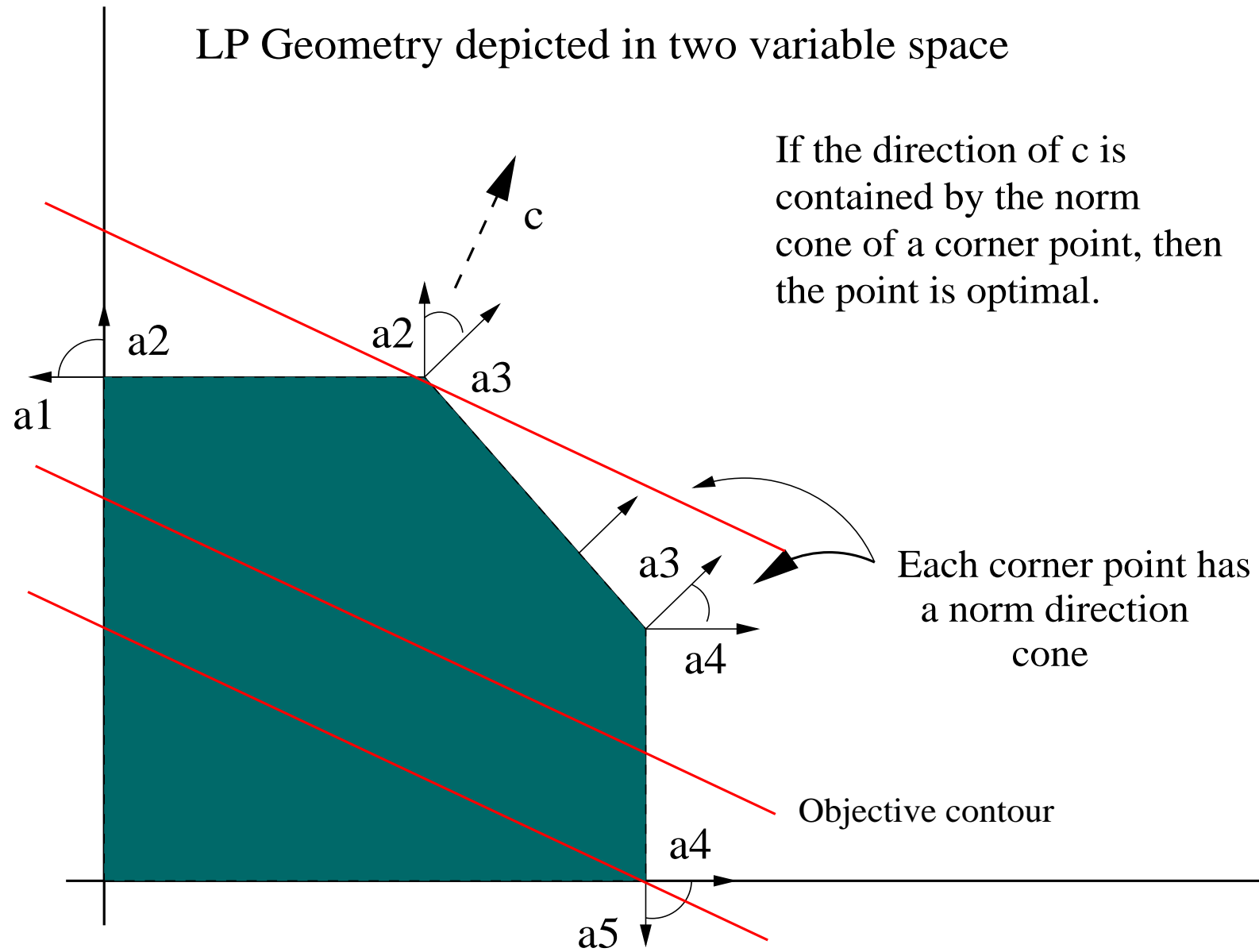
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Geometry of linear programming

Consider

$$\begin{array}{llll} \text{maximize} & x_1 & +2x_2 & \\ \text{subject to} & x_1 & & \leq 1 \\ & & x_2 & \leq 1 \\ & x_1 & +x_2 & \leq 1.5 \\ & x_1, & x_2 & \geq 0. \end{array}$$

LP Geometry depicted in two variable space



- **solution** (decision, point): any specification of values for all decision variables, regardless of whether it is a desirable or even allowable choice
- **feasible solution**: a solution for which all the constraints are satisfied.
- **feasible region** (constraint set, feasible set): the collection of all feasible solution
- **interior, boundary, face**
- **extreme or corner or vertex point**
- **objective function contour** (iso-profit, iso-cost line)
- **optimal solution**: a feasible solution that has the most favorable value of the objective function
- **optimal objective value**: the value of the objective function evaluated at an optimal solution
- **active constraint** (binding constraint)

Formal definition of face and extreme point

Let P be a polyhedron in \mathcal{R}^n , F is a face of P if and only if there is a vector \mathbf{b} for which F is the set of points attaining $\max \{\mathbf{b}^T \mathbf{y} : \mathbf{y} \in P\}$ provided this maximum is finite.

A polyhedron has only finite many faces; each face is a nonempty polyhedron.

A vector $\mathbf{y} \in P$ is an extreme point or a vertex of P if \mathbf{y} is not a convex combination of more than one distinct points.

Theory of Linear Programming

All LP problems fall into one of three classes:

- Problem is **infeasible**: Feasible region is empty.
- Problem is **unbounded**: Feasible region is unbounded towards the optimizing direction.
- Problem is **feasible and bounded**. In this case:
 - there exists an **optimal solution or optimizer**.
 - There may be a **unique** optimizer or **multiple** optimizers.
 - All optimizers are on a **face** of the feasible region.
 - There is always at least one **corner (extreme)** optimizer if the face has a corner.
 - If a corner point is not **worse** than all its **adjacent or neighboring** corners, then it is optimal.

History

George B. Dantzig's **Simplex Method** for linear programming stands as one of the most significant algorithmic achievements of the 20th century. It is now over 50 years old and still going strong.

The basic idea of the simplex method is to confine the search to **corner points** of the feasible region (of which there are only **finitely** many) in a most intelligent way. In contrast, **interior-point methods** will move in the interior of the feasible region, hoping to by-pass many **corner points** on the boundary of the region.

The key for the simplex method is to make computers **see** corner points; and the key for interior-point methods is to **stay** in the interior of the feasible region.

From geometry to algebra

- How to make computer recognize a **corner point**?
- How to make computer tell that two corners are **neighboring**?
- How to make computer **terminate** and declare optimality?
- How to make computer identify a better **neighboring corner**?

Linear Programming Standard Form

$$\text{minimize } c_1x_1 + c_2x_2 + \dots + c_nx_n$$

$$\text{subject to } a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1,$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2,$$

...

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m,$$

$$x_j \geq 0, \quad j = 1, 2, \dots, n.$$

$$(LP) \quad \text{minimize } \mathbf{c}^T \mathbf{x}$$

$$\text{subject to } A\mathbf{x} = \mathbf{b},$$

$$\mathbf{x} \geq \mathbf{0}.$$

Reduction to the Standard Form

- Eliminating "free" variables: substitute with the difference of two nonnegative variables

$$x = x^+ - x^-, \quad x^+, x^- \geq 0.$$

- Eliminating inequalities: add slack variable

$$\mathbf{a}^T \mathbf{x} \leq b \implies \mathbf{a}^T \mathbf{x} + s = b, \quad s \geq 0$$

$$\mathbf{a}^T \mathbf{x} \geq b \implies \mathbf{a}^T \mathbf{x} - s = b, \quad s \geq 0$$

- Eliminating upper bounds: move them to constraints

$$x \leq 3 \implies x + s = 3, \quad s \geq 0$$

- Eliminating nonzero **lower bounds**: shift the decision variables

$$x \geq 3 \implies x := x - 3$$

- Change $\max \mathbf{c}^T \mathbf{x}$ to $\min -\mathbf{c}^T \mathbf{x}$

The LP example in standard form

$$\begin{array}{llllll} \text{minimize} & -x_1 & -2x_2 & & & \\ \text{subject to} & x_1 & & +x_3 & & = 1 \\ & & x_2 & & +x_4 & = 1 \\ & x_1 & +x_2 & & & +x_5 = 1.5 \\ & x_1, & x_2, & x_3, & x_4, & x_5 \geq 0. \end{array}$$

Basic and Basic Feasible Solution

In the LP standard form, select m linearly independent columns, denoted by the index set B , from A . Solve

$$A_B \mathbf{x}_B = \mathbf{b}$$

for the m -vector \mathbf{x}_B . By setting the variables, \mathbf{x}_N , of \mathbf{x} corresponding to the remaining columns of A equal to zero, we obtain a solution \mathbf{x} such that $A\mathbf{x} = \mathbf{b}$.

Then, \mathbf{x} is said to be a **basic solution** to (LP) with respect to **basis** A_B . The components of \mathbf{x}_B are called **basic variables** and those of \mathbf{x}_N are **nonbasic variables**.

Two basic solutions are **adjacent** if they differ by exactly one basic (or nonbasic) variable.

If a basic solution $\mathbf{x}_B \geq \mathbf{0}$, then \mathbf{x} is called a **basic feasible solution**.

If one or more components in \mathbf{x}_B has value zero, \mathbf{x} is said to be **degenerate**.

Geometry vs Algebra

Theorem 1 *Consider the polyhedron in the standard LP form. Then, a basic feasible solution and a corner point are equivalent; the former is algebraic and the latter is geometric.*

For the LP example:

Basis	3,4,5	1,4,5	3,4,1	3,2,5	3,4,2	1,2,3	1,2,4	1,2,5
Feasible?	✓	✓		✓		✓	✓	
x_1, x_2	0, 0	1, 0	1.5, 0	0, 1	0, 1.5	.5, 1	1, .5	1, 1

Theorem 2 (*LP fundamental theorem in Algebraic form*) Given (LP) and (LD) where A has full row rank m ,

- i) if there is a feasible solution, there is a *basic feasible solution*;
- ii) if there is an optimal solution, there is an *optimal basic solution*.

The simplex method is to proceed from one **BFS** (a corner point of the feasible region) to an **adjacent or neighboring** one, in such a way as to continuously improve the value of the objective function.

Rounding to a basic feasible solution for LP

1. Start at any feasible solution x^0 and, without loss of generality, assume $x^0 > 0$, and let $k = 0$ and $A^0 = A$.
2. Find any $A^k d = 0$, $d \neq 0$, and let $x^{k+1} = x^k + \alpha d$ where α is chosen such as $x^{k+1} \geq 0$ and at least one of x^{k+1} equals 0.
3. Eliminate the the variable(s) in x^{k+1} and column(s) in A^k corresponding to $x_j^{k+1} = 0$, and let the new matrix be A^{k+1} .
4. Return to step 2.

Neighboring Basic Solutions

Two basic solutions are **neighboring or adjacent** if they differ by **exactly one** basic (or nonbasic) variable.

A basic feasible solution is optimal if no **"better"** neighboring basic feasible solution exists.

Optimality test

Consider BFS: $x_1 = 0, x_2 = 0, x_3 = 1, x_4 = 1, x_5 = 1.5$. It's **not optimal** to the problem

$$\begin{array}{llllll} \text{minimize} & -x_1 & -2x_2 & & & \\ \text{subject to} & x_1 & & +x_3 & & = 1 \\ & & x_2 & & +x_4 & = 1 \\ & x_1 & +x_2 & & & +x_5 = 1.5 \\ & x_1, & x_2, & x_3, & x_4, & x_5 \geq 0; \end{array}$$

but it's **optimal** to the problem

$$\begin{array}{llllll}
 \text{minimize} & x_1 & +2x_2 & & & \\
 \text{subject to} & x_1 & & +x_3 & & = 1 \\
 & & x_2 & & +x_4 & = 1 \\
 & x_1 & +x_2 & & & +x_5 = 1.5 \\
 & x_1, & x_2, & x_3, & x_4, & x_5 \geq 0.
 \end{array}$$

At a BFS, when the objective coefficients to all the basic variables are **zero** and those to all the non-basic variables are **non-negative**, then the BFS is optimal.

LP canonical form

A standard-form LP is said to be in the **canonical form** at a basic feasible solution, if

- the **objective coefficients** to all the basic variables are **zero**, that is, the objective function contains only non-basic variables;
- The **constraint matrix** for the basic variables form an **identity matrix** (with some permutation if necessary).

It's easy to see whether or not the current BFS is **optimal or not**, if the LP is in **canonical form**.

Is a way to transform the LP problem to an **equivalent** LP in canonical form, that is, its **feasible set and optimal solution set** remain the same sets of the original LP?

Transforming to canonical form

Suppose the basis of a BFS is A_B and the rest is A_N , one can transform the **equality** constraint to

$$A_B^{-1} A \mathbf{x} = A_B^{-1} \mathbf{b}, \quad (\text{so that } \mathbf{x}_B = A_B^{-1} \mathbf{b} - A_B^{-1} A_N \mathbf{x}_N)$$

that is, to express \mathbf{x}_B by \mathbf{x}_N , the **degree of freedom** variables. Then, the **objective function** becomes:

$$\begin{aligned} \mathbf{c}^T \mathbf{x} &= \mathbf{c}_B^T \mathbf{x}_B + \mathbf{c}_N^T \mathbf{x}_N &= \mathbf{c}_B^T A_B^{-1} \mathbf{b} - \mathbf{c}_B^T A_B^{-1} A_N \mathbf{x}_N + \mathbf{c}_N^T \mathbf{x}_N \\ & &= \mathbf{c}_B^T A_B^{-1} \mathbf{b} + (\mathbf{c}_N^T - \mathbf{c}_B^T A_B^{-1} A_N) \mathbf{x}_N. \end{aligned}$$

The transformed problem is then in **canonical form** with respect to the BFS A_B ; and it's **equivalent** to the original LP, that is, its **feasible set and optimal solution set** are remaining the same.

An equivalent LP

By ignoring the **constant term** in the objective function, we have an **equivalent** intermediate LP problem in canonical form:

$$\begin{aligned} & \text{minimize} && \mathbf{r}^T \mathbf{x} \\ & \text{subject to} && \bar{A}\mathbf{x} = \bar{\mathbf{b}}, \\ & && \mathbf{x} \geq \mathbf{0}, \end{aligned}$$

where

$$\mathbf{r}_B = \mathbf{0}, \quad \mathbf{r}_N = \mathbf{c}_N - A_N^T (A_B^{-1})^T \mathbf{c}_B, \quad \bar{A} = A_B^{-1} A, \quad \bar{\mathbf{b}} = A_B^{-1} \mathbf{b}.$$

Optimality test again

Vector $\mathbf{r} \in R^n$

$$\mathbf{r} = \mathbf{c} - \bar{A}^T \mathbf{c}_B = \mathbf{c} - A^T (A_B^{-1})^T \mathbf{c}_B$$

is called the **reduced cost coefficient** vector.

Often, we represent $(A_B^{-1})^T \mathbf{c}_B$ by **shadow price vector** \mathbf{y} , that is,

$$A_B^T \mathbf{y} = \mathbf{c}_B, \quad (\text{so that } \mathbf{r} = \mathbf{c} - A^T \mathbf{y}).$$

Theorem 3 If $\mathbf{r}_N \geq \mathbf{0}$ (equivalently $\mathbf{r} \geq \mathbf{0}$) at a BFS with basic variable set B , then the BFS \mathbf{x} is an **optimal basic solution** and A_B is an **optimal basis**.

In the LP Example, let the basic variable set $B = \{1, 2, 3\}$ so that

$$A_B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

and

$$A_B^{-1} = \begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}$$

$$\mathbf{y}^T = (0, -1, -1) \quad \text{and} \quad \mathbf{r}^T = (0, 0, 0, 1, 1)$$

The corresponding optimal corner solution is $(x_1, x_2) = (0.5, 1)$ in the **original** problem.

Canonical Tableau

The intermediate canonical form data can be organized in a tableau:

B	\mathbf{r}^T	$-\mathbf{c}_B^T \bar{\mathbf{b}}$
basis indices	\bar{A}	$\bar{\mathbf{b}}$

The **up-right corner quantity** represents negative of the objective value at the **current** basic solution.

With the initial basic variable set $B = \{3, 4, 5\}$ for the example, the canonical tableau is

B	-1	-2	0	0	0	0
3	1	0	1	0	0	1
4	0	1	0	1	0	1
5	1	1	0	0	1	$\frac{3}{2}$

Moving to a "better" neighboring corner

As we know before, the basic feasible solution given by the canonical form is optimal if the reduced-cost vector $\mathbf{r} \geq \mathbf{0}$.

What to do if it is not optimal?

An effort can be made to find a better **neighboring** basic feasible solution, that is, a **new** BFS that differs from the **current** one by exactly one **new basic variable**, as long as the reduced cost coefficient of the **entering variable** is negative.

Changing basis for the LP example

With the initial basic variable set $B = \{3, 4, 5\}$:

$$\begin{array}{c|ccccc|c}
 \mathbf{B} & -1 & -2 & 0 & 0 & 0 & 0 \\
 \hline
 3 & 1 & 0 & 1 & 0 & 0 & 1 \\
 4 & 0 & 1 & 0 & 1 & 0 & 1 \\
 5 & 1 & 1 & 0 & 0 & 1 & \frac{3}{2}
 \end{array}$$

Select the entering basic variable x_1 with $r_1 = -1 < 0$, and consider

$$\begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \frac{3}{2} \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} x_1.$$

The question is: How much we can increase x_1 while the current basic variables remain feasible (non-negative)? The **minimum ratio test (MRT)**.

Minimum Ratio Test

- Select the entering variable x_e with its reduced cost $r_e < 0$;
- If column $\bar{A}_{.e} \leq 0$, then **Unbounded**
- **minimum ratio test (MRT):**

$$\theta = \min\left\{\frac{\bar{b}_i}{\bar{A}_{ie}} : \bar{A}_{ie} > 0\right\}.$$

How much can the new basic variable be increased

What is this θ ? It is the largest amount that x_e can be increased before one (or more) of the current basic variables x_i decreases to zero.

For the moment, assume that the minimum ratio in the MRT is attained by exactly one basic variable index, o . Then, x_o is called the **out-going** basic variable:

$$x_o = \bar{b}_o - \bar{a}_{oe}\theta = 0$$

$$x_i = \bar{b}_i - \bar{a}_{ie}\theta > 0 \quad \forall i \neq o.$$

That is,

$$x_o \longleftarrow x_e$$

in the basic variable set.

Tie Breaking

If the MRT does not result in a single index, but rather in a set of two or more indices for which the minimum ratio is attained, we select one of these indices as out-going arbitrarily.

Again, when x_e reaches θ we have generated a corner point of the feasible region that is adjacent to the one associated with the index set B . The new basic variable set associated with this “new” corner point is

$$o \leftarrow e$$

.

But the new basic feasible solution is **degenerate**, one of its basic variable has value 0. Pretending it's $\epsilon > 0$ but arbitrarily small and continue the transformation process to the new canonical form.

The Simplex Algorithm

0. **Initialize** with a minimization problem in feasible canonical form with respect to a basic index set B . Let N denote the complementary index set.

1. **Test for termination.** First find

$$r_e = \min_{j \in N} \{r_j\}.$$

If $r_e \geq 0$, stop. The solution is optimal. Otherwise determine whether the column of $\bar{A}_{.e}$ contains a positive entry. If not, the objective function is unbounded below. Terminate. Let x_e be the entering basic variable.

2. **Determine the outgoing** Execute the MRT to determine the outgoing variable x_o .

3. **Update basis.** Update B and A_B and transform the problem to canonical form, and return to Step 1.

Transform to new canonical form: Pivoting

The passage from one canonical form to the next can be carried out by an algebraic process called **pivoting**. In a nutshell, a **pivot** on the nonzero (in our case, positive) **pivot element** \bar{a}_{oe} amounts to

- (a) expressing x_e in terms of all the non-basic variables;
- (b) replacing x_e in all other formula by using the expression.

In terms of \bar{A} , this has the effect of making the column of x_e look like the column of an identity matrix with **1** in the o^{th} row and zero elsewhere.

Pivoting: Gauss-Jordan Elimination Process

Here is what happens to the current table.

1. All the entries in **row** o are divided by the pivot element \bar{a}_{oe} . (This produces a 1 (one) in the column of x_s .)
2. For all $i \neq o$, all other entries are modified according to the rule

$$\bar{a}_{ij} \longleftarrow \bar{a}_{ij} - \frac{\bar{a}_{oj}}{\bar{a}_{oe}} \bar{a}_{ie}.$$

(When $j = e$, the new entry is just 0 (zero).)

The right-hand side and the objective function row are modified in just the same way.

The initial canonical form of the example

Choose $e = 2$ and MRT would decide $o = 4$ with $\theta = 1$:

B	-1	-2	0	0	0		MRT
3	1	0	1	0	0	1	∞
4	0	1	0	1	0	1	1
5	1	1	0	0	1	$\frac{3}{2}$	$\frac{3}{2}$

If necessary, dividing the pivoting row by the pivot value to make the pivot element equal to 1.

Transform to new canonical form by pivoting

“objective row := objective row + 2*pivot row”:

B	-1	0	0	2	0	2
3	1	0	1	0	0	1
2	0	1	0	1	0	1
5	1	1	0	0	1	$\frac{3}{2}$

“bottom row := bottom row - pivot row”:

B	-1	0	0	2	0	2
3	1	0	1	0	0	1
2	0	1	0	1	0	1
5	1	0	0	-1	1	$\frac{1}{2}$

This is the new canonical form!

Let's keep going

Choose $e = 1$ and MRT would decide $o = 5$ with $\theta = 1/2$:

B	-1	0	0	2	0	2	MRT
3	1	0	1	0	0	1	1
2	0	1	0	1	0	1	∞
5	1	0	0	-1	1	$\frac{1}{2}$	$\frac{1}{2}$

Transform to new canonical form by pivoting

“objective row := objective row + pivot row”:

B	0	0	0	1	1	$2\frac{1}{2}$
3	1	0	1	0	0	1
2	0	1	0	1	0	1
1	1	0	0	-1	1	$\frac{1}{2}$

“second row := second row - pivot row”:

B	0	0	0	1	1	$2\frac{1}{2}$
3	0	0	1	1	-1	$\frac{1}{2}$
2	0	1	0	1	0	1
1	1	0	0	-1	1	$\frac{1}{2}$

Stop: it's optimal:)

Summary of the Simplex Method in Tableau

0. **Initialize** with a minimization problem in feasible canonical form with respect to a basic index set B . Let N denote the complementary index set.

1. **Test for termination.** First find

$$r_e = \min_{j \in N} \{r_j\}.$$

If $r_e \geq 0$, stop. The solution is optimal. Otherwise determine whether the column of x_e contains a positive entry. If not, the objective function is unbounded below. Terminate.

2. **Minimum ratio test.** Execute the MRT to determine the pivot row o and the pivot element, \bar{a}_{oe} .

3. **Pivot step.** Pivot on \bar{a}_{oe} and modify the definitions of B and N . Return to Step 1.