Dual Interpretations and Duality Applications

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This week: Appendix B, Chapters 2.2, 2.6, 3.1-3.6, 6.3-6.4

Production Problem I

$$\max\ \mathbf{p}^T\mathbf{x} \quad \text{s.t.} \quad A\mathbf{x} \le \mathbf{r}, \quad \mathbf{x} \ge \mathbf{0}$$

where

- $\mathbf{p}$: profit margin vector
- A: resources consumption rate matrix
- $\mathbf{r}$: available resource vector
- $\mathbf{x}$: production level decision vector

Production Problem II: Liquidation Pricing

- $\mathbf{y}$: the fair price vector
- $A^T\mathbf{y} \geq \mathbf{p}$: competitiveness
- $\mathbf{y} \geq 0$: positivity
- $\min\ \mathbf{r}^T\mathbf{y}$: minimize the total liquidation cost

$$
\text{Primal}: \quad \begin{array}{lllll}
\text{maximize} & x_1 & +2x_2 & & \\
\text{subject to} & x_1 & & \leq 1 & \\
& & x_2 & \leq 1 & \\
& x_1 & +x_2 & \leq 1.5 & \\
& x_1, & x_2 & \geq 0. &
\end{array}
$$

$$
\text{Dual}: \quad \begin{array}{llllll}
\text{minimize} & y_1 & +y_2 & +1.5y_3 & \\
\text{subject to} & y_1 & & +y_3 & \geq 1 \\
& & y_2 & +y_3 & \geq 2 \\
& y_1, & y_2, & y_3 & \geq 0.
\end{array}
$$

Optimal Value Function and Shadow Prices

$$z(\mathbf{b}) = \quad \text{minimize} \quad \mathbf{c}^T\mathbf{x}$$
$$\text{subject to} \quad A\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0}.$$

Suppose a new right-hand-vector $\mathbf{b}^+$ such that

$$b_k^+ = b_k + \delta \quad \text{and} \quad b_i^+ = b_i,\ \forall i \neq k.$$

Then, the optimal dual solution $\mathbf{y}^*$ has a property

$$y_k^* = (z(\mathbf{b}^+) - z(\mathbf{b}))/\delta$$

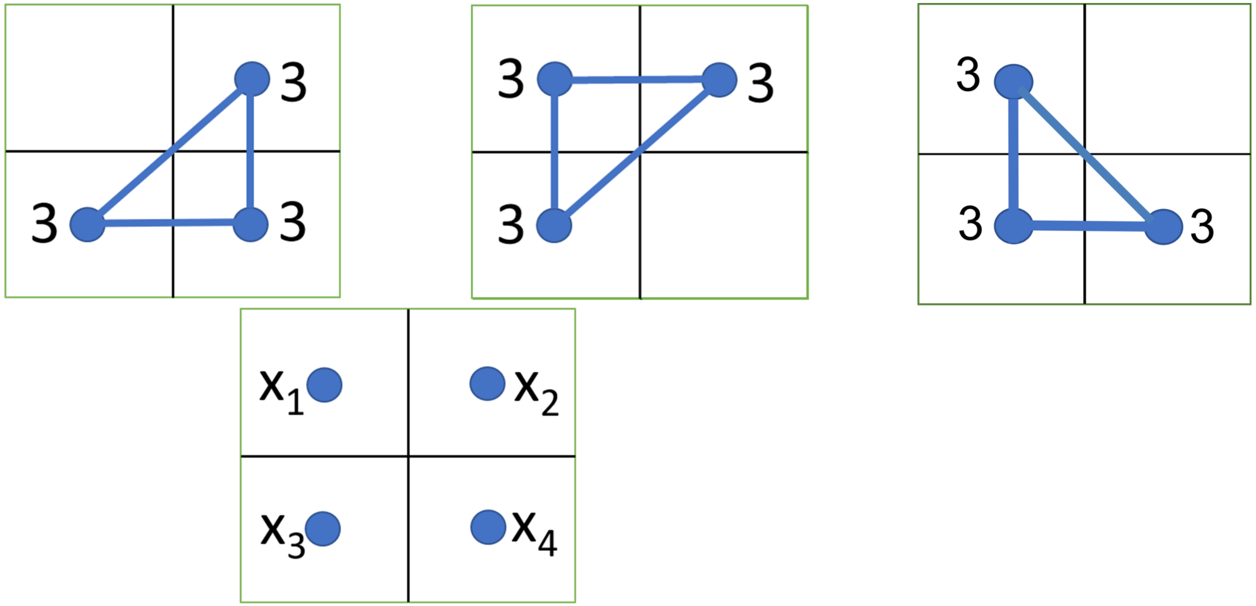
as long as $\mathbf{y}^*$ remains the dual optimal solution for $\mathbf{b}^+$, because

$$z(\mathbf{b}^+) = (\mathbf{b}^+)^T\mathbf{y}^* = z(\mathbf{b}) + \delta \cdot y_k^*.$$

Thus, the optimal dual value is the rate of the net change of the optimal objective value over the net change of an entry of the right-hand-vector resources, i.e.,

$$\nabla z(\mathbf{b}) = \mathbf{y}^*.$$

Application in the Wassestein Barycenter Problem



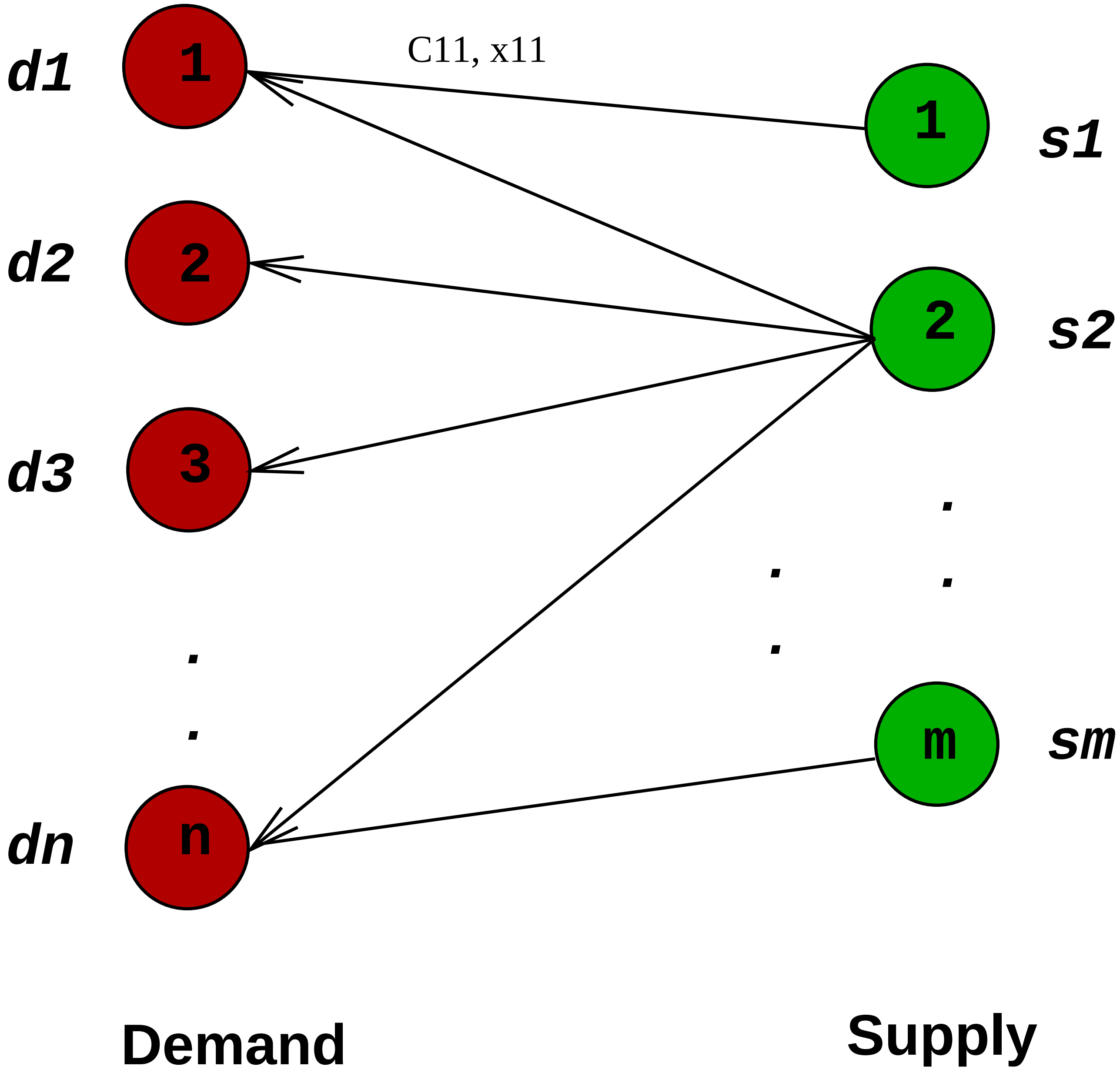
Find distribution of $x_i, i = 1, 2, 3, 4$ to minimize

$$\begin{array}{ll} \min & WD_l(\mathbf{x}) + WD_m(\mathbf{x}) + WD_r(\mathbf{x}) \\ \text{s.t.} & x_1 + x_2 + x_3 + x_4 = 9, \quad x_i \geq 0, \ i = 1, 2, 3, 4. \end{array}$$

The objective is a nonlinear function, but its gradient vector $\nabla WD_l(\mathbf{x})$, $\nabla WD_m(\mathbf{x})$ and $\nabla WD_l(\mathbf{x})$ are shadow prices of the three sub-transportation problems –popularly used in Hierarchy Optimization.

Recall Transportation Problem

$$
\begin{array}{lll}
\min & \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} & \\
\text{s.t.} & \sum_{j=1}^{n} x_{ij} & = s_i,\ \forall i = 1,...,m \\
& \sum_{i=1}^{m} x_{ij} & = d_j,\ \forall j = 1,...,n \\
& x_{ij} & \geq 0,\ \forall i,j.
\end{array}
$$


d1
d2
d3
dn
1
2
3
n
C11, x11
1
2
m
s1
s2
sm
Demand
Supply

Transportation Dual: Economic Interpretation

$$\begin{array}{lll} \max & \sum_{i=1}^{m} s_i u_i + \sum_{j=1}^{n} d_j v_j & \\ \text{s.t.} & u_i + v_j & \leq c_{ij}, \ \forall i, j. \end{array}$$

u_i: supply site unit price

v_i: demand site unit price

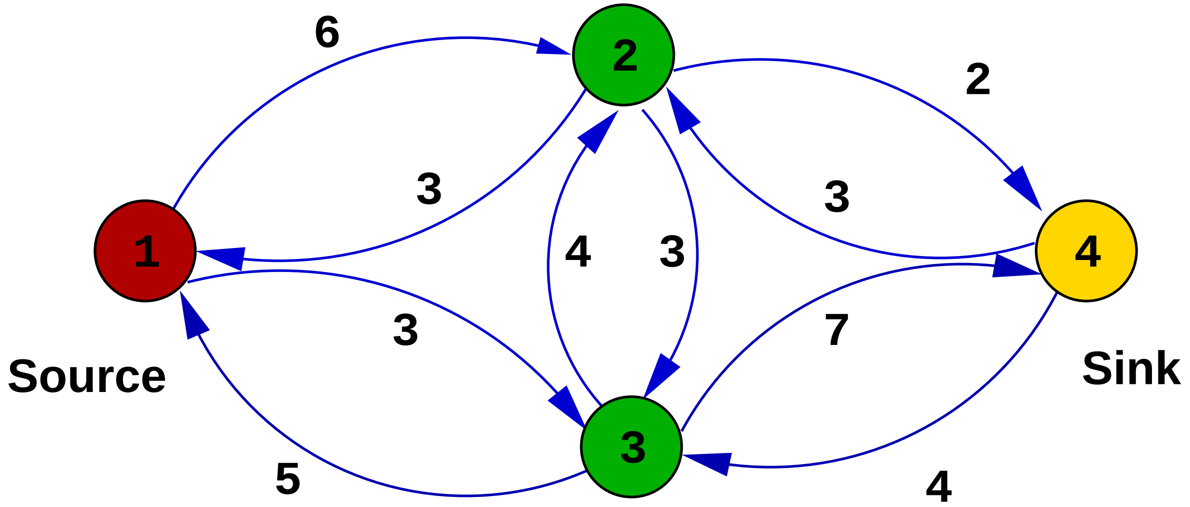
$u_i + v_j \leq c_{ij}$: competitiveness

Max-Flow and Min-Cut

Given a directed graph with nodes $1, ..., m$ and edges $\mathcal{A}$, where node 1 is called source and node m is called the sink, and each edge (i, j) has a flow rate capacity k_{ij}. The Max-Flow problem is to find the largest possible flow rate from source to sink.

Let x_{ij} be the flow rate from node i to node j. Then the problem can be formulated as

$$\begin{array}{ll} \text{maximize} & x_{m1} \\ \text{subject to} & \sum_{j:(j,1)\in\mathcal{A}} x_{j1} - \sum_{j:(1,j)\in\mathcal{A}} x_{1j} + x_{m1} = 0, \\ & \sum_{j:(j,i)\in\mathcal{A}} x_{ji} - \sum_{j:(i,j)\in\mathcal{A}} x_{ij} = 0, \forall i = 2, ..., m-1, \\ & \sum_{j:(j,m)\in\mathcal{A}} x_{jm} - \sum_{j:(m,j)\in\mathcal{A}} x_{mj} - x_{m1} = 0, \\ & 0 \le x_{ij} \le k_{ij}, \ \forall (i,j) \in \mathcal{A}. \end{array}$$



6
2
2
3
3
1
4
4
3
3
7
Source
Sink
3
5
4

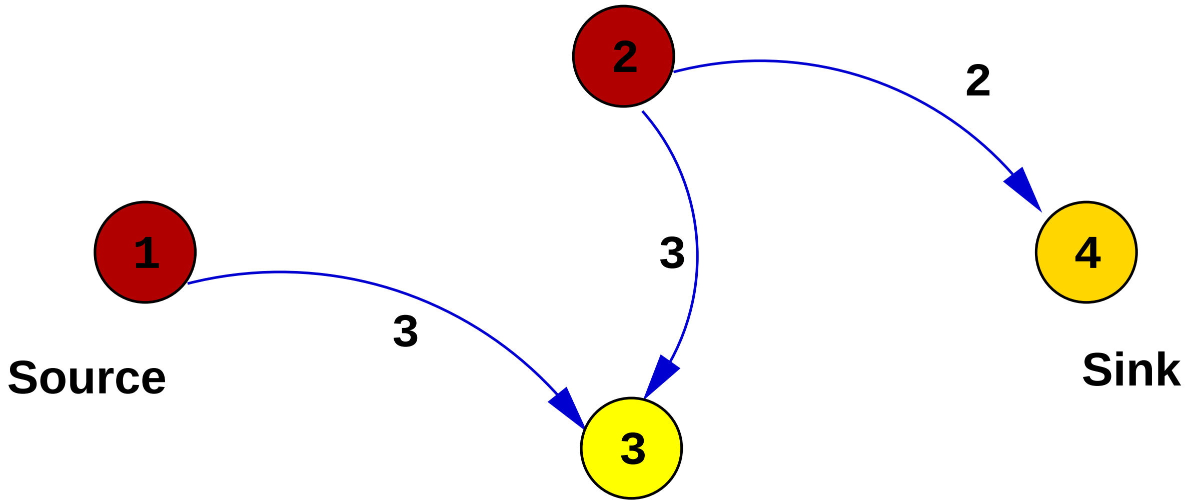
The dual of the Max-Flow problem

$$\begin{array}{ll} \text{minimize} & \sum_{(i,j)\in\mathcal{A}} k_{ij} z_{ij} \\ \text{subject to} & -y_i + y_j + z_{ij} \geq 0, \ \forall (i,j) \in \mathcal{A}, \\ & y_1 - y_m = 1, \\ & z_{ij} \geq 0, \ \forall (i,j) \in A. \end{array}$$

y_i: node potential value. At an optimal solution has property $y_1 = 1, \ y_m = 0$ and for all other i:

$$y_i = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{if } i \notin S \end{cases}$$

This problem is called the Min-Cut problem.


2
1
3
4
3
3
2
Source
Sink

The Dual of the Reinforcement Learning LP

Recall the cost-to-go value of the reinforcement learning LP problem:

$$
\begin{array}{lrcl}
\text{maximize}_{\mathbf{y}} & \sum_{i=1}^m y_i & & \\
\text{subject to} & y_1 - \gamma \mathbf{p}_j^T \mathbf{y} & \leq & c_j,\ j \in \mathcal{A}_1 \\
 & & \ldots & \\
 & y_i - \gamma \mathbf{p}_j^T \mathbf{y} & \leq & c_j,\ j \in \mathcal{A}_i \\
 & & \ldots & \\
 & y_m - \gamma \mathbf{p}_j^T \mathbf{y} & \leq & c_j,\ j \in \mathcal{A}_m.
\end{array}
$$

$$
\begin{array}{lccccc}
\text{minimize}_{\mathbf{x}} & \sum_{j\in\mathcal{A}_1} c_j x_j + & \ldots & + \sum_{j\in\mathcal{A}_m} c_j x_j & & \\
\text{subject to} & \sum_{j\in\mathcal{A}_1} (\mathbf{e}_1 - \gamma \mathbf{p}_j) x_j + & \ldots & + \sum_{j\in\mathcal{A}_m} (\mathbf{e}_m - \gamma \mathbf{p}_j) x_j & = & \mathbf{e}, \\
 & \ldots & x_j & \ldots & \geq & 0,\ \forall j,
\end{array}
$$

where $\mathbf{e}_i$ is the unit vector with 1 at the ith position and 0 everywhere else.

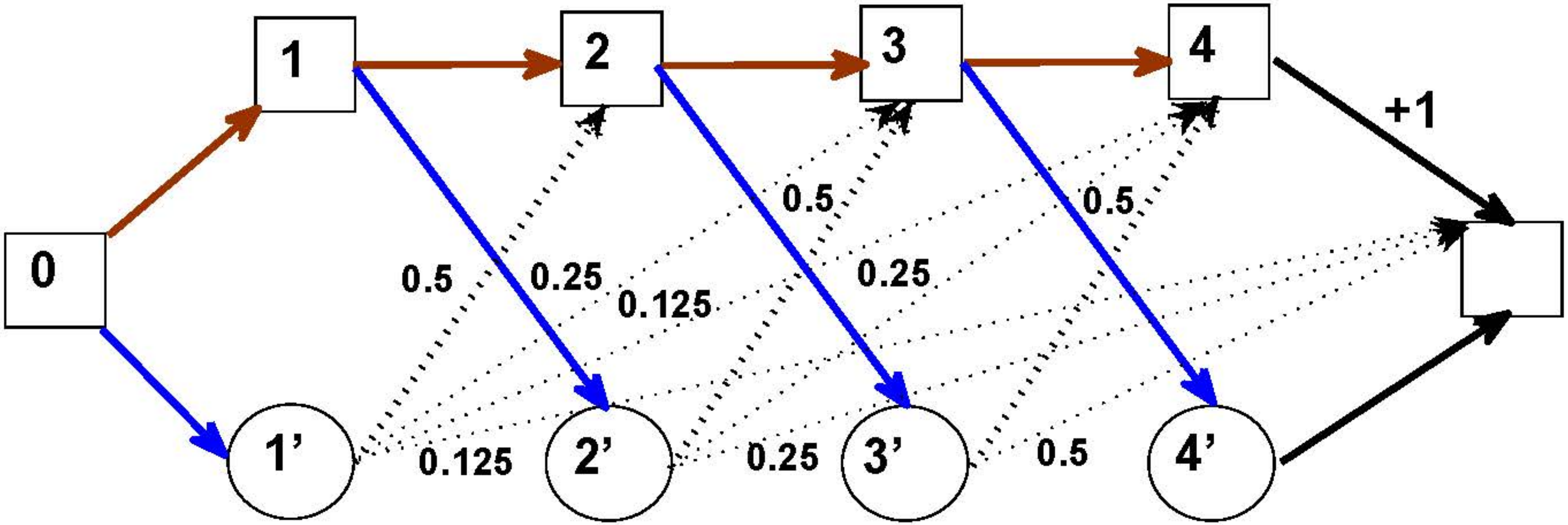
Interpretation of the Dual of the RL-LP

Variable x_j, $j \in \mathcal{A}_i$, is the state-action frequency or called flux, or the expected present value of the number of times that an individual is in state i and takes state-action j.

Thus, solving the problem entails choosing a state-action frequencies/fluxes that minimizes the expected present value of total costs for the infinite horizon, where the RHS is $(1; 1; 1; 1; 1; 1)$:

x:	(0_1)	(0_2)	(1_1)	(1_2)	(2_1)	(2_2)	(3_1)	(3_2)	(4_1)	(5_1)	**b**
c:	0	0	0	0	0	0	0	0	1	0	
(0)	1	1	0	0	0	0	0	0	0	0	1
(1)	$-\gamma$	0	1	1	0	0	0	0	0	0	1
(2)	0	$-\gamma/2$	$-\gamma$	0	1	1	0	0	0	0	1
(3)	0	$-\gamma/4$	0	$-\gamma/2$	$-\gamma$	0	1	1	0	0	1
(4)	0	$-\gamma/8$	0	$-\gamma/4$	0	$-\gamma/2$	$-\gamma$	0	1	0	1
(5)	0	$-\gamma/8$	0	$-\gamma/4$	0	$-\gamma/2$	0	$-\gamma$	$-\gamma$	$1-\gamma$	1

where state 5 is the absorbing state that has a infinite loops to itself.



The optimal dual solution is

$$x^*_{01} = 1,\; x^*_{11} = 1 + \gamma,\; x^*_{21} = 1 + \gamma + \gamma^2,\; x^*_{32} = 1 + \gamma + \gamma^2 + \gamma^3,\; x^*_{41} = 1,$$
$$x^*_{51} = \frac{1+2\gamma+\gamma^2+\gamma^3+\gamma^4}{1-\gamma}.$$

The Maze Runner Example: Complementarity Condition

The LP optimal Cost-to-Go values are $y_1^* = 0, y_1^* = 0, y_2^* = 0, y_3^* = 0, y_4^* = 1$:

$$\begin{array}{lrl}
\text{maximize}_{\mathbf{y}} & y_0 + y_1 + y_2 + y_3 + y_4 + y_5 & \\
\text{subject to} & y_0 - \gamma y_1 & \leq 0, \ (x_{01}^* = 1) \\
& y_0 - \gamma(0.5y_2 + 0.25y_3 + 0.125y_4) & \leq 0, \ (x_{02}^* = 0) \\
& y_1 - \gamma y_2 & \leq 0, \ (x_{11}^* = 1 + \gamma) \\
& y_1 - \gamma(0.5y_3 + 0.25y_4) & \leq 0, \ (x_{12}^* = 0) \\
& y_2 - \gamma y_3 & \leq 0, \ (x_{21}^* = 1 + \gamma + \gamma^2) \\
& y_2 - \gamma(0.5y_4) & \leq 0, \ (x_{22}^* = 0) \\
& y_3 - \gamma y_4 & \leq 0, \ (x_{31}^* = 0) \\
& y_3 & \leq 0, \ (x_{32}^* = 1 + \gamma + \gamma^2 + \gamma^3) \\
& y_4 - \gamma y_5 & \leq 1, \ (x_{41}^* = 1) \\
& y_5 - \gamma y_5 & = 0. \ (x_{51}^* = \frac{1+2\gamma+\gamma^2+\gamma^3+\gamma^4}{1-\gamma})
\end{array}$$

Dual of Information Markets

$$\begin{array}{lrl} \max & \pi^T \mathbf{x} - z & \\ \text{s.t.} & A\mathbf{x} - \mathbf{e} \cdot z & \leq \mathbf{0}, \\ & \mathbf{x} & \leq \mathbf{q}, \\ & \mathbf{x} & \geq 0. \end{array}$$

$\pi^T \mathbf{x}$: the optimistic amount can be collected.

z: the worst-case amount need to pay to the winning bids.

$$\begin{array}{lrl} \min & \mathbf{q}^T \mathbf{y} & \\ \text{s.t.} & A^T \mathbf{p} + \mathbf{y} & \geq \pi, \\ & \mathbf{e}^T \mathbf{p} & = 1, \\ & (\mathbf{p}, \mathbf{y}) & \geq 0. \end{array}$$

$\mathbf{p}$ represents the state prices or probability distributions.

Dual Interpretation: Regression using Important Data Samples

Note that

$$y_j = \max\{0,\ \pi_j - \mathbf{a}_j^T \mathbf{p}\},\ \forall j.$$

so that

$$\begin{array}{lrl} \min & \sum_j \max\{0,\ \pi_j - \mathbf{a}_j^T \mathbf{p}\} & \\ \text{s.t.} & \mathbf{e}^T \mathbf{p} & = 1, \\ & \mathbf{p} & \geq 0. \end{array}$$

The $\max\{0,\ \cdot\}$ is called ReLu function in AI.

Dual Interpretation: Find the probability estimations such that low-bids are automatically uncounted/removed.

Strictly Complementarity Condition in Information Markets

$x_j > 0$	$\mathbf{a}_j^T\mathbf{p} + y_j = \pi_j$ and $y_j \geq 0$ so that $\mathbf{a}_j^T\mathbf{p} \leq \pi_j$
$0 < x_j < q_j$	$y_j = 0$ so that $\mathbf{a}_j^T\mathbf{p} = \pi_j$
$x_j = q_j$	$y_j > 0$ so that $\mathbf{a}_j^T\mathbf{p} < \pi_j$
$x_j = 0$	$\mathbf{a}_j^T\mathbf{p} + y_j > \pi_j$ and $y_j = 0$ so that $\mathbf{a}_j^T\mathbf{p} > \pi_j$

The price is Fair:

$$\mathbf{p}^T(A\mathbf{x} - \mathbf{e} \cdot z) = 0 \quad \text{implies} \quad \mathbf{p}^T A\mathbf{x} = \mathbf{p}^T\mathbf{e} \cdot z = z;$$

that is, the worst case cost equals the worth of total shares. Moreover, if a lower bid wins the auction, so does the higher bid on any same type of bids.

World Cup Information Market Result

Order:	$\#1$	$\#2$	$\#3$	$\#4$	$\#5$	State Price
Argentina	1	0	1	1	0	0.2
Brazil	1	0	0	1	1	0.35
Italy	1	0	1	1	0	0.2
Germany	0	1	0	1	1	0.25
France	0	0	1	0	0	0
Bidding Price:π	0.75	0.35	0.4	0.95	0.75	
Quantity limit:$\mathbf{q}$	10	5	10	10	5	
Order fill:$\mathbf{x}^*$	5	5	5	0	5	

Question: How to make the dual prices unique and the market online?

Online Information Markets

$$\begin{array}{lrl} \max & \pi^T \mathbf{x} - z + u(\mathbf{s}) & \\ \text{s.t.} & A\mathbf{x} - \mathbf{e} \cdot z + \mathbf{s} & = \mathbf{0}, \\ & \mathbf{x} & \leq \mathbf{q}, \\ & \mathbf{x},\ \mathbf{s} & \geq 0. \end{array}$$

$u(\mathbf{s})$: a value function for the market organizer on slack shares.

If $u(\cdot)$ is a strictly concave function, then the state price vector is unique.

Online Market: solve the convex problem sequentially - SCPM.

General Resource Allocation Example

	order 1($t=1$)	order 2($t=2$)	.....	Inventory($\mathbf{b}$)
Price(π_t)	\$100	\$30	...	
Decision	x_1	x_2	...	
Pants	1	0	...	100
Shoes	1	0	...	50
T-shirts	0	1	...	500
Jacket	0	0	...	200
Socks	1	1	...	1000

SCPM for Online Resource Allocation

$$
\begin{array}{ll}
\text{maximize}_{\mathbf{x}} & \sum_{j=1}^{n} \pi_j x_j \\
\text{s.t.} & \sum_{j=1}^{n} a_{ij} x_j \le b_i, \ \forall\, i = 1, 2, ..., m, \\
& 0 \le x_j \le 1, \ \forall\, j = 1, ..., n.
\end{array}
$$

Approach 1 (SCPM):

$$
\begin{array}{ll}
\text{maximize}_{x_k, \mathbf{s}} & \pi_k x_k + u(\mathbf{s}) \\
\text{s.t.} & a_{ik} x_k + s_i = b_i - \sum_{j=1}^{k-1} a_{ik} \bar{x}_k, \ \forall\, i = 1, 2, ..., m, \\
& 0 \le x_k \le 1, \\
& s_i \ge 0, \ \forall\, i = 1, ..., m.
\end{array}
$$