Applications of Duality

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Production Problem I

$$\max\ \mathbf{p}^T\mathbf{x} \quad \text{s.t.} \quad A\mathbf{x} \le \mathbf{r}, \quad \mathbf{x} \ge \mathbf{0}$$

where

- $\mathbf{p}$: profit margin vector
- A: resources consumption rate matrix
- $\mathbf{r}$: available resource vector
- $\mathbf{x}$: production level decision vector

Production Problem II: Liquidation Pricing

- $\mathbf{y}$: the fair price vector
- $A^T\mathbf{y} \geq \mathbf{p}$: competitiveness
- $\mathbf{y} \geq 0$: positivity
- $\min\ \mathbf{r}^T\mathbf{y}$: minimize the total liquidation cost

$$\text{Primal}: \quad \begin{array}{lcccc} \text{maximize} & x_1 & +2x_2 & & \\ \text{subject to} & x_1 & & \le 1 & \\ & & x_2 & \le 1 & \\ & x_1 & +x_2 & \le 1.5 & \\ & x_1, & x_2 & \ge 0. & \end{array}$$

$$\text{Dual}: \quad \begin{array}{lcccc} \text{minimize} & y_1 & +y_2 & +1.5y_3 & \\ \text{subject to} & y_1 & & +y_3 & \ge 1 \\ & & y_2 & +y_3 & \ge 2 \\ & y_1, & y_2, & y_3 & \ge 0. \end{array}$$

Optimal Value Function

For fixed matrix A and objective coefficient vector $\mathbf{c}$, the optimal value is a function of right-hand-side vector $\mathbf{b}$:

$$\begin{array}{rll} z(\mathbf{b}) = & \text{minimize} & \mathbf{c}^T\mathbf{x} \\ & \text{subject to} & A\mathbf{x} = \mathbf{b}, \\ & & \mathbf{x} \geq \mathbf{0}. \end{array}$$

Theorem: $z(\mathbf{b})$ is a convex function in $\mathbf{b}$, that is, for any $0 \leq \alpha \leq 1$

$$z(\alpha\mathbf{b}^1 + (1-\alpha)\mathbf{b}^2) \leq \alpha z(\mathbf{b}^1) + (1-\alpha) z(\mathbf{b}^2).$$

Shadow Prices of the Optimal Value

Suppose a new right-hand-vector $\mathbf{b}^+$ such that

$$b_k^+ = b_k + \delta \quad \text{and} \quad b_i^+ = b_i, \; \forall i \neq k.$$

Then, the optimal dual solution $\mathbf{y}^*$ has a property

$$y_k^* = \frac{z(\mathbf{b}^+) - z(\mathbf{b})}{\delta}$$

as long as $\mathbf{y}^*$ remains the dual optimal solution for $\mathbf{b}^+$, since

$$z(\mathbf{b}^+) = (\mathbf{b}^+)^T \mathbf{y}^* = z(\mathbf{b}) + \delta \cdot y_k^*.$$

Thus, the optimal dual solution is the shadow price vector of the right-hand-vector vector, or the rate of the net change of the optimal objective value over the net change of an entry of the right-hand-vector vector.

Combinatorial Auction I

Given m potential states that are mutually exclusive and exactly one of them will be realized at the maturity.

An order is a bet on one or a combination of states, with a price limit (the maximum price the participant is willing to pay for one unit of the order) and a quantity limit (the maximum number of units the participant is willing to accept).

A contract on an order is a paper agreement so that on maturity it is worth a notional \$$1$ dollar if the order includes the winning state and worth \$$0$ otherwise.

A Call Auction is organized on the n submitted orders.

Combinatorial Auction II: an order

The jth order is given as $(\mathbf{a}_j \in R^m_+, \; \pi_j \in R_+, \; q_j \in R_+)$: $\mathbf{a}_j$ is the betting indication vector where each entry is either 1 or 0

$$\mathbf{a}_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \dots \\ a_{mj} \end{pmatrix},$$

where 1 is winning state and 0 is non-winning state; π_j is the price limit for one unit of a contract, and q_j is the maximum number of units or shares the better like to own.

Combinatorial Auction III: Organizer's problem

$$\begin{array}{rrl} \max & \pi^T \mathbf{x} - s & \\ \text{s.t.} & A\mathbf{x} - \mathbf{e} \cdot s & \leq 0, \\ & \mathbf{x} & \leq \mathbf{q}, \\ & \mathbf{x} & \geq 0. \end{array}$$

$\pi^T \mathbf{x}$: the amount can be collected.

s: the worst-case amount need to pay back.

Combinatorial Auction IV: The dual

$$
\begin{array}{lrl}
\min & \mathbf{q}^T\mathbf{y} & \\
\text{s.t.} & A^T\mathbf{p}+\mathbf{y} & \geq \pi, \\
 & \mathbf{e}^T\mathbf{p} & = 1, \\
 & (\mathbf{p},\mathbf{y}) & \geq 0.
\end{array}
$$

$\mathbf{p}$ represents the state price and What is $\mathbf{y}$?

Combinatorial Auction V: Strict Complementarity

$x_j > 0$	$\mathbf{a}_j^T\mathbf{p} + y_j = \pi_j$ and $y_j \ge 0$ so that $\mathbf{a}_j^T\mathbf{p} \le \pi_j$
$0 < x_j < q_j$	$y_j = 0$ so that $\mathbf{a}_j^T\mathbf{p} = \pi_j$
$x_j = q_j$	$y_j > 0$ so that $\mathbf{a}_j^T\mathbf{p} < \pi_j$
$x_j = 0$	$\mathbf{a}_j^T\mathbf{p} + y_j > \pi_j$ and $y_j = 0$ so that $\mathbf{a}_j^T\mathbf{p} > \pi_j$

The price is Fair:

$$\mathbf{p}^T(A\mathbf{x} - \mathbf{e}\cdot s) = 0 \quad \text{implies} \quad \mathbf{p}^T A\mathbf{x} = \mathbf{p}^T\mathbf{e}\cdot s = s.$$

Transportation Dual

$$\begin{array}{lll} \min & \sum_{i=1}^m \sum_{j=1}^n c_{ij} x_{ij} & \\ \text{s.t.} & \sum_{j=1}^n x_{ij} & = s_i, \ \forall i = 1, ..., m \\ & \sum_{i=1}^m x_{ij} & = d_j, \ \forall j = 1, ..., n \\ & x_{ij} & \geq 0, \ \forall i, j. \end{array}$$

$$\begin{array}{lll} \max & \sum_{i=1}^m s_i u_i + \sum_{j=1}^n d_j v_j & \\ \text{s.t.} & u_i + v_j & \leq c_{ij}, \ \forall i, j. \end{array}$$

u_i: supply site unit price

v_i: demand site unit price

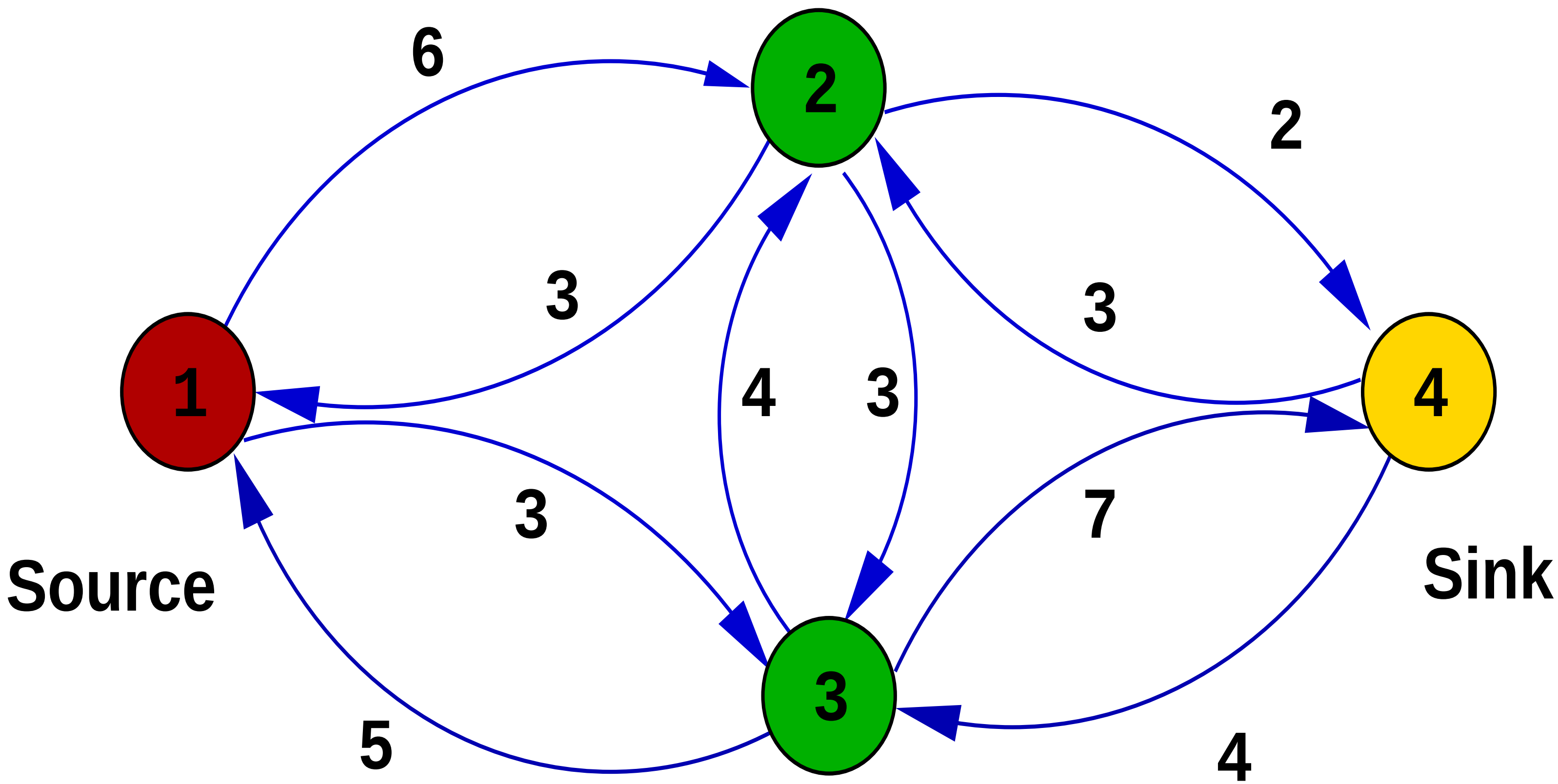
$u_i + v_j \leq c_{ij}$: competitiveness

Max-Flow and Min-Cut

Given a directed graph with nodes $1, ..., m$ and edges $\mathcal{A}$, where node 1 is called source and node m is called the sink, and each edge (i, j) has a flow rate capacity k_{ij}. The Max-Flow problem is to find the largest possible flow rate from source to sink.

Let x_{ij} be the flow rate from node i to node j. Then the problem can be formulated as

$$
\begin{array}{ll}
\text{maximize} & x_{m1} \\
\text{subject to} & \sum_{j:(j,1)\in\mathcal{A}} x_{j1} - \sum_{j:(1,j)\in\mathcal{A}} x_{1j} + x_{m1} = 0, \\
& \sum_{j:(j,i)\in\mathcal{A}} x_{ji} - \sum_{j:(i,j)\in\mathcal{A}} x_{ij} = 0, \forall i = 2, ..., m-1, \\
& \sum_{j:(j,m)\in\mathcal{A}} x_{jm} - \sum_{j:(m,j)\in\mathcal{A}} x_{mj} - x_{m1} = 0, \\
& 0 \le x_{ij} \le k_{ij}, \ \forall (i,j) \in \mathcal{A}.
\end{array}
$$


6
2
3
2
3
1
4
4
3
3
7
Source
Sink
3
5
4

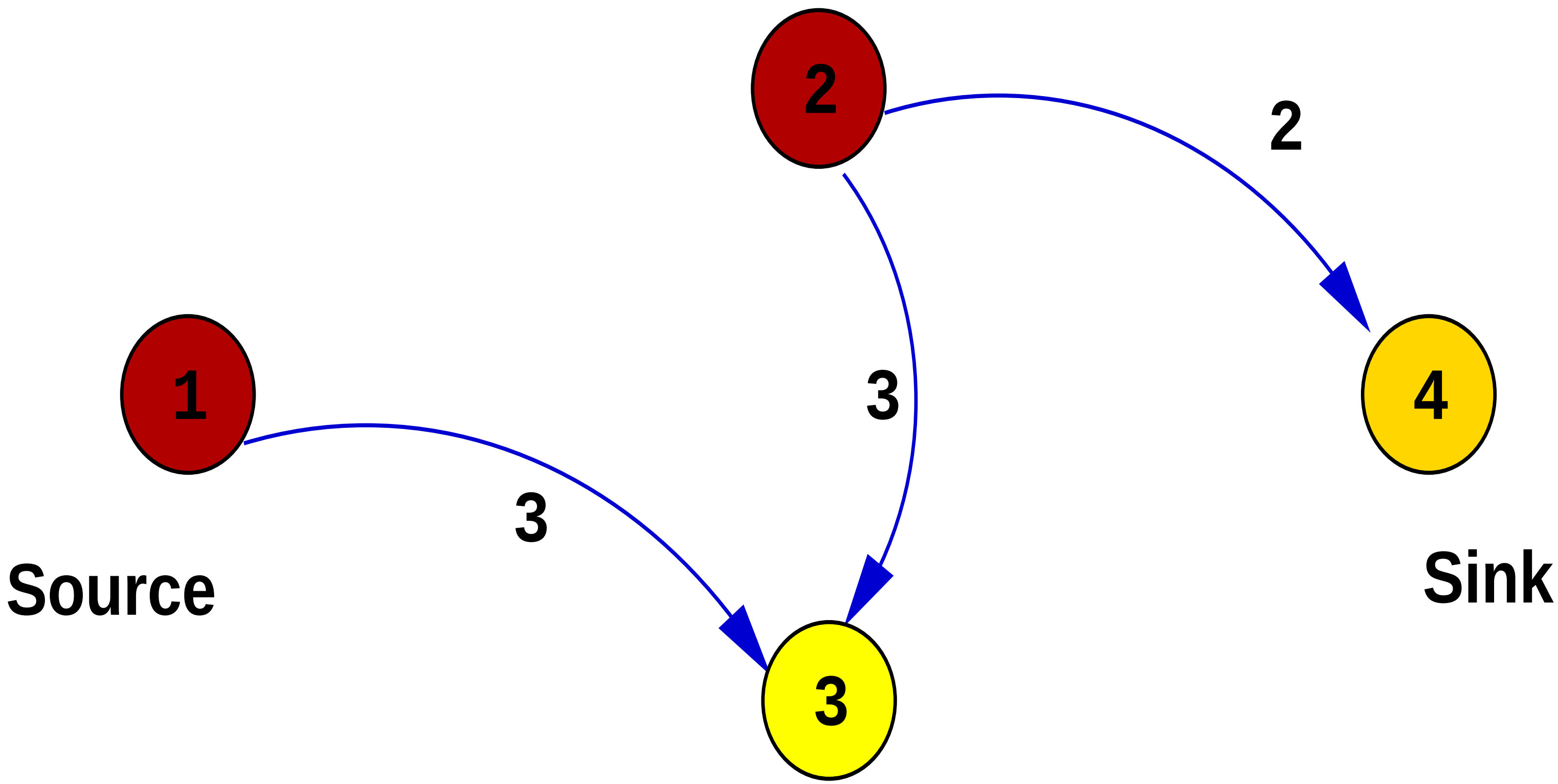
The dual of the Max-Flow problem

$$\begin{array}{ll} \text{minimize} & \sum_{(i,j)\in\mathcal{A}} k_{ij} z_{ij} \\ \text{subject to} & -y_i + y_j + z_{ij} \geq 0, \ \forall (i,j) \in \mathcal{A}, \\ & y_1 - y_m = 1, \\ & z_{ij} \geq 0, \ \forall (i,j) \in A. \end{array}$$

y_i: node potential value. At an optimal solution has property $y_1 = 1, \ y_m = 0$ and for all other i:

$$y_i = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{if } i \notin S \end{cases}$$

This problem is called the Min-Cut problem.


2
2
3
1
4
3
Source
Sink
3

Multi-Firm LP Alliance I

Consider a finite set I of firms each of whom has operations that have representations as linear programs. Suppose the linear program representing the operations of firm i in I entails choosing an n-column vector $\mathbf{x} \geq \mathbf{0}$ of activity levels that maximize the firm's profit

$$\mathbf{c}^T\mathbf{x}$$

subject to the constraint that its consumption $A\mathbf{x}$ of resources minorizes its available resource vector $\mathbf{b}^i$, that is,

$$A\mathbf{x} \leq \mathbf{b}^i.$$

Multi-Firm LP Alliance II

An alliance is a subset of the firms. If an alliance S pools its resource vectors, the linear program that S faces is that of choosing an n-column vector $\mathbf{x} \geq \mathbf{0}$ that maximizes the profit $\mathbf{c}^T\mathbf{x}$ that S earns subject to its resource constraint

$$A\mathbf{x} \leq \mathbf{b}^S = \sum_{i \in S} \mathbf{b}^i.$$

Let V^S be the resulting maximum profit of S. The grand alliance is the set I of all firms.

$$\begin{array}{rlll} V^S := & \max & \mathbf{c}^T\mathbf{x} & \\ & \text{s.t.} & A\mathbf{x} & \leq \sum_{i \in S} \mathbf{b}^i, \\ & & \mathbf{x} & \geq \mathbf{0}, \end{array}$$

Multi-Firm LP Alliance III: Core

Core is the set of payment vector $\mathbf{z} = (z_1, \dots, z_{|I|})$ to each company such that

$$\sum_{i \in I} z_i = V^I$$

and

$$\sum_{i \in S} z_i \geq V^S, \; \forall S \subset I.$$

Theorem 1 *For each optimal dual price vector for the linear program of the grand alliance, allocating each firm the value of its resource vector at those prices yields a profit allocation vector in the core.*

Robust Optimization I

Consider a linear program

$$\begin{array}{ll} \text{minimize} & (\mathbf{c} + C\mathbf{u})^T\mathbf{x} \\ \text{subject to} & A\mathbf{x} = \mathbf{b}, \\ & \mathbf{x} \geq \mathbf{0}, \end{array}$$

where $\mathbf{u} \geq \mathbf{0}$ and $\mathbf{u} \leq \mathbf{e}$ is a state of Nature and beyond decision maker's control.

Robust Model:

$$\begin{array}{ll} \text{minimize} & \max_{\{\mathbf{u} \geq \mathbf{z},\ \mathbf{u} \leq \mathbf{e}\}} (\mathbf{c} + C\mathbf{u})^T\mathbf{x} \\ \text{subject to} & A\mathbf{x} = \mathbf{b}, \\ & \mathbf{x} \geq \mathbf{0}. \end{array}$$

Robust Optimization II

Nature's (primal) problem:

$$\begin{array}{rrcl} \text{maximize}_{\mathbf{u}} & \mathbf{c}^T\mathbf{x} + \mathbf{x}^T C\mathbf{u} & & \\ \text{subject to} & \mathbf{u} & \leq & \mathbf{e}, \\ & \mathbf{u} & \geq & \mathbf{0}. \end{array}$$

Dual of Nature's problem:

$$\begin{array}{rrcl} \text{minimize}_{\mathbf{y}} & \mathbf{c}^T\mathbf{x} + \mathbf{e}^T\mathbf{y} & & \\ \text{subject to} & \mathbf{y} & \geq & C^T\mathbf{x}, \\ & \mathbf{y} & \geq & \mathbf{0}. \end{array}$$

Robust Optimization III

Decision Maker's Robust Model:

$$
\begin{aligned}
\text{minimize}_{\mathbf{x},\mathbf{y}} \quad & \mathbf{c}^T\mathbf{x} + \mathbf{e}^T\mathbf{y} \\
\text{subject to} \quad & \mathbf{y} \ \geq \ C^T\mathbf{x}, \\
& A\mathbf{x} \ = \ \mathbf{b}, \\
& \mathbf{x}, \ \mathbf{y} \ \geq \ \mathbf{0}.
\end{aligned}
$$

Two-Person Zero-Sum Game

Let P be the payoff matrix of a two-person, ”column” and ”row”, zero-sum game.

$$P = \begin{pmatrix} +3 & -1 & -4 \\ -3 & +1 & +4 \end{pmatrix}$$

Players usually use randomized strategies in such a game. A randomized strategy is a vector of probabilities, each associated with a particular decision.

Nash Equilibrium

In a Nash Equilibrium, if your (column) strategy is a pure strategy (one where you always play a single action), the expected payout for the (dominating) action that you are playing should be greater than or equal to the expected payout for any other action. If you are playing a randomized strategy, the expected payout for each action included in your strategy should be the same (if one were lower, you won't want to ever choose that action) and these payouts should be greater than or equal to the actions that aren't part of your strategy.

LP formulation of Nash Equilibrium

"Column" strategy:

$$\begin{array}{ll} \max & v \\ s.t. & v\mathbf{e} \le P\mathbf{x} \\ & \mathbf{e}^T\mathbf{x} = 1 \\ & \mathbf{x} \ge \mathbf{0}. \end{array}$$

"Row" strategy:

$$\begin{array}{ll} \min & u \\ s.t. & u\mathbf{e} \ge P^T\mathbf{y} \\ & \mathbf{e}^T\mathbf{y} = 1 \\ & \mathbf{y} \ge \mathbf{0}. \end{array}$$

They are dual to each other.

Fisher's equilibrium price

Player $i \in B$'s optimization problem for given prices p_j, $j \in G$.

$$\begin{array}{rl} \text{maximize} & \mathbf{u}_i^T \mathbf{x}_i := \sum_{j \in G} u_{ij} x_{ij} \\ \text{subject to} & \mathbf{p}^T \mathbf{x}_i := \sum_{j \in G} p_j x_{ij} \le w_i, \\ & x_{ij} \ge 0, \quad \forall j, \end{array}$$

Without losing generality, assume that the amount of each good is 1. The equilinitum price vector is the one that for all $j \in G$

$$\sum_{i \in B} x(\mathbf{p})_{ij} = 1$$

Equilibrium price conditions

Player $i \in B$'s dual problem for given prices p_j, $j \in G$.

$$\begin{array}{ll} \text{minimize} & w_i y_i \\ \text{subject to} & \mathbf{p} y_i \geq \mathbf{u}_i, \; y_i \geq 0 \end{array}$$

The necessary and sufficient conditions for an equilibrium point $\mathbf{x}_i, \mathbf{p}$ are:

$$\begin{array}{ll} \mathbf{p}^T \mathbf{x}_i \leq w_i, \; \mathbf{x}_i \geq \mathbf{0}, & \forall i, \\ p_j y_i \geq u_{ij}, \; y_i \geq 0, & \forall i, j, \\ \mathbf{u}_i^T \mathbf{x}_i = w_i y_i, & \forall i, \\ \sum_i x_{ij} = 1, & \forall j. \end{array}$$

Equilibrium price conditions continued

These conditions can be represented by

$$\begin{array}{ll} \sum_i p_i \le \sum_i w_i, \ \mathbf{x}_i \ge \mathbf{0}, & \forall i, \\ \frac{\mathbf{u}_i^T \mathbf{x}_i}{w_i} \cdot p_j \ge u_{ij}, & \forall i, j, \\ \sum_i x_{ij} = 1, & \forall j. \end{array}$$

since from the second inequality (after multiplying $x_i j$ to both sides and take sum over j) we have

$$\mathbf{p}^T \mathbf{x}_i \ge w_i, \ \forall i.$$

Then, from the rest conditions

$$\sum_i w_i \ge \sum_i p_i = \sum_i \mathbf{p}^T \mathbf{x}_i \ge \sum_i w_i.$$

Thus, these conditions imply $\mathbf{p}^T \mathbf{x}_i = w_i, \ \forall i$.

Equilibrium price property

If u_{ij} has at least one positive coefficient for every j, then we must have $p_j > 0$ for every j at every equilibrium. Morevoer, The second inequality can be rewritten as

$$\log(\mathbf{u}_i^T \mathbf{x}_i) + \log(p_j) \geq \log(w_i) + \log(u_{ij}), \ \forall i, j, \ u_{ij} > 0.$$

The function on the left is (strictly) concave in $\mathbf{x}_i$ and p_j. Thus,

Theorem 2 *The equilibrium set of the Fisher Market is convex, and the equilbrium price vector is unique.*

Example of Fisher's equilibrium price

Buyer $1, 2$'s optimization problems for given prices p_x, p_y.

$$\begin{array}{ll} \text{maximize} & 2x_1 + y_1 \\ \text{subject to} & p_x \cdot x_1 + p_y \cdot y_1 \leq 5, \\ & x_1, y_1 \geq 0; \\ \text{maximize} & 3x_2 + y_2 \\ \text{subject to} & p_x \cdot x_2 + p_y \cdot y_2 \leq 8, \\ & x_2, y_2 \geq 0. \end{array}$$

$$p_x = \frac{26}{3}, \quad p_y = \frac{13}{3}$$

$$x_1 = \frac{1}{13},\ y_1 = 1,\ x_2 = \frac{12}{13},\ y_2 = 0$$

Formulation 7: Arrow-Debreu's Exchange Market

Producer exchange goods to maximize their individual utility functions. The equilibrium price is an assignment of prices to goods so as when every producer sells his/her good and buys a maximal bundle of goods then the market clears, meaning that all goods are sold and bought. Thus, producer/buyer $i \in P$'s optimization problem for given prices p_j, $j \in P$.

$$\begin{array}{ll} \text{maximize} & \mathbf{u}_i^T \mathbf{x}_i := \sum_{j \in P} u_{ij} x_{ij} \\ \text{subject to} & \mathbf{p}^T \mathbf{x}_i := \sum_{j \in P} p_j x_{ij} \le p_i, \\ & x_{ij} \ge 0, \quad \forall j, \end{array}$$

The equilibrium price vector is the one that for all $j \in G$

$$\sum_{i \in P} x(p)_{ij} = 1.$$

Example of Arrow-Debreu's Model

Producer $1(x), 2(y)$'s optimization problems for given prices p_x, p_y.

$$
\begin{array}{ll}
\text{maximize} & 2x_1 + y_1 \\
\text{subject to} & p_x \cdot x_1 + p_y \cdot y_1 \le p_x, \\
 & x_1, y_1 \ge 0; \\
\text{maximize} & 3x_2 + y_2 \\
\text{subject to} & p_x \cdot x_2 + p_y \cdot y_2 \le p_y, \\
 & x_2, y_2 \ge 0.
\end{array}
$$

$$p_x = 2, \quad p_y = 1$$

$$x_1 = \frac{1}{2},\ y_1 = 1,\ x_2 = \frac{1}{2},\ y_2 = 0$$

Equilibrium price of the Arrow-Debreu market

Similary, the equilibrium conditions of the Arrow-Debreu market are

$$\begin{array}{rl} p_i > 0,\ \mathbf{x}_i \geq \mathbf{0}, & \forall i, \\ \frac{\mathbf{u}_i^T \mathbf{x}_i}{p_i} \cdot p_j \geq u_{ij}, & \forall i, j, \\ \sum_i x_{ij} = 1, & \forall j. \end{array}$$

Morevoer, The second inequality can be rewritten as

$$\log(\mathbf{u}_i^T \mathbf{x}_i) + \log(p_j) - \log(p_i) \geq \log(u_{ij}),\ \forall i, j,\ u_{ij} > 0.$$

Treat $\log(p_i)$ as variable y_i, then it becomes

$$\log(\mathbf{u}_i^T \mathbf{x}_i) + y_j - y_i \geq \log(u_{ij}),\ \forall i, j,\ u_{ij} > 0.$$

The function on the left is concave in $\mathbf{x}_i$ and y_j. Thus,

Theorem 3 *The equilibrium set of the Arrow-Debreu Market is convex in the logarithmic of price.*