# Mathematical Preliminaries

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Chapters 1 and Appendixes A,B.1-B.2,C.1

## Real $n$-Space; Euclidean Space

- $\mathcal{R}, \quad \mathcal{R}_+, \quad \text{int}\,\mathcal{R}_+$
- $\mathcal{R}^n, \quad \mathcal{R}^n_+, \quad \text{int}\,\mathcal{R}^n_+$
- $\mathbf{x} \ge \mathbf{y}$ means $x_j \ge y_j$ for $j = 1, 2, ..., n$
- $\mathbf{0}$: all zero vector; and $\mathbf{e}$: all one vector
- Inner-Product:

$$\mathbf{x} \bullet \mathbf{y} := \mathbf{x}^T\mathbf{y} = \sum_{j=1}^{n} x_j y_j$$

- Norm: $\|\mathbf{x}\|_2 = \sqrt{\mathbf{x}^T\mathbf{x}}, \quad \|\mathbf{x}\|_\infty = \max\{|x_1|, |x_2|, ..., |x_n|\}, \quad \|\mathbf{x}\|_p = \left(\sum_{j=1}^n |x_j|^p\right)^{1/p}$
- The dual of the $p$ norm, denoted by $\|.\|^*$, is the $q$ norm, where $\frac{1}{p} + \frac{1}{q} = 1$
- Column vector:

$$\mathbf{x} = (x_1; x_2; \ldots; x_n)$$

and row vector:

$$\mathbf{x} = (x_1, x_2, \ldots, x_n)$$

- A set of vectors $\mathbf{a}_1, ..., \mathbf{a}_m$ is said to be linearly dependent if there are scalars $\lambda_1, ..., \lambda_m$, not all zero, such that the linear combination

$$\sum_{i=1}^{m} \lambda_i \mathbf{a}_i = \mathbf{0}$$

- A linearly independent set of vectors that span $R^n$ is a basis.

- For a sequence $\mathbf{x}^k \in R^n$, $k = 0, 1, ...$, we say it is a contraction sequence if there is an $\mathbf{x}^* \in R^n$ and a scalar constant $0 < \gamma < 1$ such that

$$\|\mathbf{x}^{k+1} - \mathbf{x}^*\| \leq \gamma \|\mathbf{x}^k - \mathbf{x}^*\|, \ \forall k \geq 0.$$

# Matrices

- $A \in \mathcal{R}^{m\times n}$; $\mathbf{a}_{i.}$, the $i$th row vector; $\mathbf{a}_{.j}$, the $j$th column vector; $a_{ij}$, the $i,j$th entry
- $\mathbf{0}$: all zero matrix, and $I$: the identity matrix
- The null space $\mathcal{N}(A)$, the row space $\mathcal{R}(A^T)$, and they are orthogonal.
- $\det(A)$, $\text{tr}(A)$: the sum of the diagonal entries of $A$
- Inner Product:

$$A \bullet B = \text{tr} A^T B = \sum_{i,j} a_{ij} b_{ij}$$

- The operator norm of matrix $A$:

$$\|A\|^2 := \max_{\mathbf{0} \neq \mathbf{x} \in \mathcal{R}^n} \frac{\|A\mathbf{x}\|^2}{\|\mathbf{x}\|^2}$$

  The Frobenius norm of matrix $A$:

$$\|A\|_f^2 := A \bullet A = \sum_{i,j} a_{ij}^2$$

- Sometimes we use $X = \text{diag}(\mathbf{x})$
- Eigenvalues and eigenvectors

$$A\mathbf{v} = \lambda \cdot \mathbf{v}$$

- Perron-Frobenius Theorem: a real square matrix with positive entries has a unique largest real eigenvalue and the corresponding eigenvector has strictly positive components.
- Stochastic Matrices: $A \geq \mathbf{0}$ with $\mathbf{e}^T A = \mathbf{e}^T$ (Column-Stochastic), or $A\mathbf{e} = \mathbf{e}$ (Row-Stochastic), or Doubly-Stochastic if both. It has a unique largest real eigenvalue $1$ and corresponding non-negative right or left eigenvector.

## Symmetric Matrices

- $\mathcal{S}^n$
- The Frobenius norm:

$$\|X\|_f = \sqrt{\text{tr} X^T X} = \sqrt{X \bullet X}$$

- Positive Definite (PD): $Q \succ \mathbf{0}$ iff $\mathbf{x}^T Q \mathbf{x} > 0, \quad \text{for all} \quad \mathbf{x} \neq \mathbf{0}$. The sum of PD matrices is PD.
- Positive Semidefinite (PSD): $Q \succeq \mathbf{0}$ iff $\mathbf{x}^T Q \mathbf{x} \geq 0, \quad \text{for all} \quad \mathbf{x}$. The sum of PSD matrices is PSD.
- PSD matrices: $\mathcal{S}^n_+, \quad \text{int}\, \mathcal{S}^n_+$ is the set of all positive definite matrices.

## Known Inequalities

- Cauchy-Schwarz: given $\mathbf{x}, \mathbf{y} \in \mathcal{R}^n$, $\mathbf{x}^T\mathbf{y} \le \|\mathbf{x}\|\|\mathbf{y}\|$.
- Triangle: given $\mathbf{x}, \mathbf{y} \in \mathcal{R}^n$, $\|\mathbf{x}+\mathbf{y}\| \le \|\mathbf{x}\| + \|\mathbf{y}\|$.
- Arithmetic-geometric mean: given $\mathbf{x} \in \mathcal{R}^n_+$,

$$\frac{\sum x_j}{n} \ge \left(\prod x_j\right)^{1/n}.$$

## Affine, Convex, Linear and Conic Combinations

When $\mathbf{x}$ and $\mathbf{y}$ are two distinct points in $R^n$ and $\alpha$ runs over $R$ ,

$$\{\mathbf{z} : \mathbf{z} = \alpha\mathbf{x} + (1-\alpha)\mathbf{y}\}$$

is the line connecting $\mathbf{x}$ and $\mathbf{y}$. When $0 \leq \alpha \leq 1$, it is called the convex combination of $\mathbf{x}$ and $\mathbf{y}$ and it is the line segment between $\mathbf{x}$ and $\mathbf{y}$.

$$\{\mathbf{z} : \mathbf{z} = \alpha\mathbf{x} + \beta\mathbf{y}\},$$

for multipliers $\alpha, \beta$, is the linear combination of $\mathbf{x}$ and $\mathbf{y}$, and it is the hyperplane containing origin and $\mathbf{x}$ and $\mathbf{y}$. When $\alpha \geq 0, \beta \geq 0$, it is called the conic combination...

## Convex Set

- $\Omega$ is said to be a convex set if for every $\mathbf{x}^1, \mathbf{x}^2 \in \Omega$ and every real number $\alpha \in [0, 1]$, the point $\alpha\mathbf{x}^1 + (1-\alpha)\mathbf{x}^2 \in \Omega$.
- Ball and Ellipsoid: for given $\mathbf{y} \in \mathcal{R}^n$ and positive definite matrix $Q$: $E(\mathbf{y}, Q) = \{\mathbf{x} : \ (\mathbf{x}-\mathbf{y})^T Q(\mathbf{x}-\mathbf{y}) \le 1\}$.
- The intersection of convex sets is convex, the sum-set of convex sets is convex, the scaled-set of a convext set is convex
- The convex hull of a set $\Omega$ is the intersection of all convex sets containing $\Omega$. Given column-points of $A$, the convex hull is $\{\mathbf{z} = A\mathbf{x} : \ \mathbf{e}^T\mathbf{x} = 1, \mathbf{x} \ge \mathbf{0}\}$.
  SVM Claim: two point sets are separable by a plane if any only if their convex hulls are separable.
- An extreme point in a convex set is a point that cannot be expressed as a convex combination of other two distinct points of the set.
- A set is polyhedral if it has finitely many extreme points; $\{\mathbf{x} : \ A\mathbf{x} = \mathbf{b}, \ \mathbf{x} \ge \mathbf{0}\}$ and $\{\mathbf{x} : \ A\mathbf{x} \le \mathbf{b}\}$ are convex polyhedral.

## Proof of convex set

- All solutions to the system of linear equations, $\{\mathbf{x} : \ A\mathbf{x} = \mathbf{b}\}$, form a convex set.
- All solutions to the system of linear inequalities, $\{\mathbf{x} : \ A\mathbf{x} \le \mathbf{b}\}$, form a convex set.
- All solutions to the system of linear equations and inequalities, $\{\mathbf{x} : \ A\mathbf{x} = \mathbf{b}, \ \mathbf{x} \ge \mathbf{0}\}$, form a convex set.
- Ball is a convex set: given center $\mathbf{y} \in \mathcal{R}^n$ and radius $r > 0$, $B(\mathbf{y}, r) = \{\mathbf{x} : \ \|\mathbf{x} - \mathbf{y}\| \le r\}$.
- Ellipsoid is a convex set: given center $\mathbf{y} \in \mathcal{R}^n$ and positive definite matrix $Q$, $E(\mathbf{y}, Q) = \{\mathbf{x} : \ (\mathbf{x} - \mathbf{y})^T Q(\mathbf{x} - \mathbf{y}) \le 1\}$.

## More on proof of convex set

Consider the set $B$ of all $\mathbf{b}$, for a fixed $A$, such that the set, $\{\mathbf{x}: \ A\mathbf{x} = \mathbf{b}, \ \mathbf{x} \geq \mathbf{0}\}$, is feasible.

Show that $B$ is a convex set.

Example:

$$B = \{b: \ \{(x_1, x_2): \ x_1 + x_2 = b, \ (x_1, x_2) \geq \mathbf{0}\} \text{ is feasible}\}.$$

## Cone and Convex Cone

- A set $C$ is a cone if $\mathbf{x} \in C$ implies $\alpha\mathbf{x} \in C$ for all $\alpha > 0$
- The intersection of cones is a cone
- A convex cone is a cone and also a convex set
- A pointed cone is a cone that does not contain a line
- Dual:

$$C^* := \{\mathbf{y} : \ \mathbf{x} \bullet \mathbf{y} \geq 0 \quad \text{for all} \quad \mathbf{x} \in C\}.$$

**Theorem 1** *The dual is always a closed convex cone, and the dual of the dual is the closure of convex hall of $C$.*

## Cone Examples

- Example 2.1: The $n$-dimensional non-negative orthant, $\mathcal{R}^n_+ = \{\mathbf{x} \in \mathcal{R}^n : \mathbf{x} \geq \mathbf{0}\}$, is a convex cone. The dual cone is itself.
- Example 2.2: The set of all positive semi-definite matrices in $\mathcal{S}^n$, $\mathcal{S}^n_+$, is a convex cone, called the positive semi-definite matrix cone. The dual cone is itself.
- Example 2.3: The set $\{\mathbf{x} \in \mathcal{R}^n : x_1 \geq \|\mathbf{x}_{-1}\|\}$, $\mathcal{N}^n_2$, is a convex cone in $\mathcal{R}^n$ called the second-order cone. The dual cone is itself.
- Example 2.4: The set $\{\mathbf{x} \in \mathcal{R}^n : x_1 \geq \|\mathbf{x}_{-1}\|_p\}$, $\mathcal{N}^n_p$, is a convex cone in $\mathcal{R}^n$ called the $p$-order cone with $p \geq 1$. The dual cone is the $q$-order cone with $\frac{1}{q} + \frac{1}{p} = 1$.

## Polyhedral Convex Cones

- A cone $C$ is (convex) polyhedral if $C$ can be represented by

$$C = \{\mathbf{x} : A\mathbf{x} \le \mathbf{0}\} \quad \text{or} \quad \{\mathbf{x} : \ \mathbf{x} = A\mathbf{y}, \ \mathbf{y} \ge \mathbf{0}\}$$

  for some matrix $A$. In the latter case, $K$ is generated by the columns of $A$.

- The nonnegative orthant is a polyhedral cone but the second-order cone is not polyhedral.

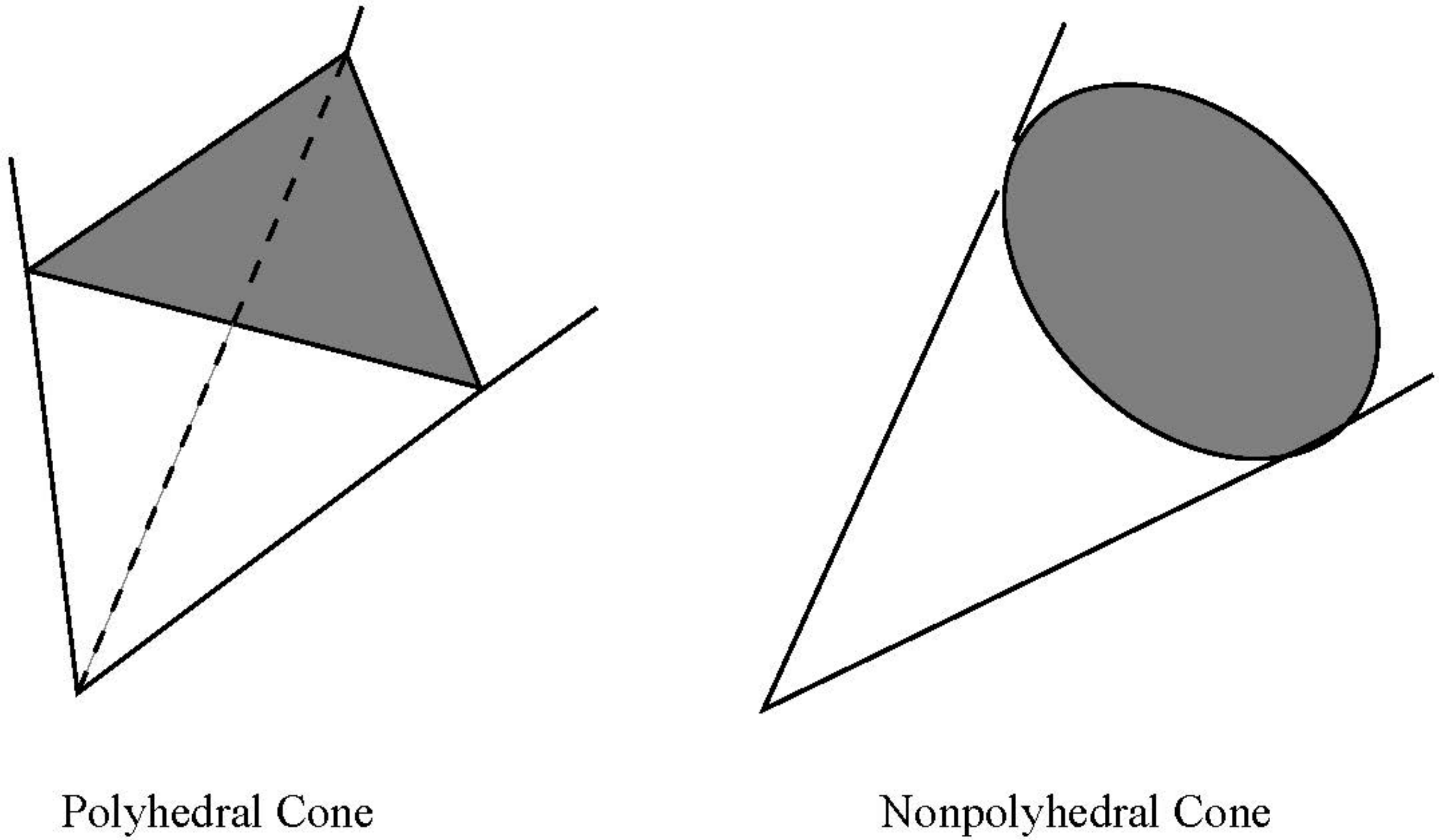


Figure 1: Polyhedral and non-polyhedral cones.

## Real Functions

- Continuous functions
- Weierstrass theorem: a continuous function $f$ defined on a compact set (bounded and closed) $\Omega \subset \mathcal{R}^n$ has a minimizer in $\Omega$.
- The gradient vector: $\nabla f(\mathbf{x}) = \{\partial f/\partial x_i\}$, for $i = 1, ..., n$.
- The Hessian matrix: $\nabla^2 f(\mathbf{x}) = \left\{ \frac{\partial^2 f}{\partial x_i \partial x_j} \right\}$ for $i = 1, ..., n;\ j = 1, ..., n.$
- Vector function: $\mathbf{f} = (f_1; f_2; ...; f_m)$
- The Jacobian matrix of $\mathbf{f}$ is

$$\nabla \mathbf{f}(x) = \begin{pmatrix} \nabla f_1(\mathbf{x}) \\ ... \\ \nabla f_m(\mathbf{x}) \end{pmatrix}.$$

- The least upper bound or supremum of $f$ over $\Omega$

$$\sup\{f(\mathbf{x}) : \mathbf{x} \in \Omega\}$$

and the greatest lower bound or infimum of $f$ over $\Omega$

$$\inf\{f(\mathbf{x}) : \mathbf{x} \in \Omega\}$$

## Convex Functions

- $f$ is a (strongly) convex function iff for $0 < \alpha < 1$,

$$f(\alpha \mathbf{x} + (1-\alpha)\mathbf{y})(<) \le \alpha f(\mathbf{x}) + (1-\alpha) f(\mathbf{y}).$$

- The sum of convex functions is a convex function; the max of convex functions is a convex function;
- The Composed function $f(\phi(\mathbf{x}))$ is convex if $\phi(\mathbf{x})$ is a convex and $f(\cdot)$ is convex&non-decreasing.
- The (lower) level set of $f$ is convex:

$$L(z) = \{\mathbf{x} : \quad f(\mathbf{x}) \le z\}.$$

- Convex set $\{(z;\mathbf{x}) : \ f(\mathbf{x}) \le z\}$ is called the epigraph of $f$.
- $tf(\mathbf{x}/t)$ is a convex function of $(t;\mathbf{x})$ for $t > 0$ if $f(\cdot)$ is a convex function; it's homogeneous with degree $1$.

# Convex Function Examples

- $\|\mathbf{x}\|_p$ for $p \geq 1$.

$$\|\alpha\mathbf{x} + (1-\alpha)\mathbf{y}\|_p \leq \|\alpha\mathbf{x}\|_p + \|(1-\alpha)\mathbf{y}\|_p \leq \alpha\|\mathbf{x}\|_p + (1-\alpha)\|\mathbf{y}\|_p,$$

  from the triangle inequality.

- Logistic function $\log(1 + e^{\mathbf{a}^T\mathbf{x}+b})$ is convex.

- $e^{x_1} + e^{x_2} + e^{x_3}$.

- $\log(e^{x_1} + e^{x_2} + e^{x_3})$: we will prove it later.

**Theorem 2** *Every local minimizer is a global minimizer in minimizing a convex objective function over a convex feasible set. If the objective is strongly convex in the feasible set, the minimizer is unique.*

**Theorem 3** *Every local minimizer is a boundary solution in minimizing a concave objective function (with non-zero gradient everywhere) over a convex feasible set. If the objective is strongly concave in the feasible set, every local minimizer must be an extreme solution.*

## Proof of convex function

Consider the minimal-objective value function of $\mathbf{b}$ for fixed $A$ and $\mathbf{c}$:

$$\begin{array}{rl} z(\mathbf{b}) := \text{minimize} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & A\mathbf{x} = \mathbf{b}, \\ & \mathbf{x} \geq \mathbf{0}. \end{array}$$

Show that $z(\mathbf{b})$ is a convex function in $\mathbf{b}$ for all feasible $\mathbf{b}$.

## Theorems on functions

Taylor's theorem or the mean-value theorem:

**Theorem 4** *Let* $f \in C^1$ *be in a region containing the line segment* $[\mathbf{x}, \mathbf{y}]$*. Then there is a* $\alpha$*,* $0 \leq \alpha \leq 1$*, such that*

$$f(\mathbf{y}) = f(\mathbf{x}) + \nabla f(\alpha \mathbf{x} + (1-\alpha)\mathbf{y})(\mathbf{y} - \mathbf{x}).$$

*Furthermore, if* $f \in C^2$ *then there is a* $\alpha$*,* $0 \leq \alpha \leq 1$*, such that*

$$f(\mathbf{y}) = f(\mathbf{x}) + \nabla f(\mathbf{x})(\mathbf{y} - \mathbf{x}) + (1/2)(\mathbf{y} - \mathbf{x})^T \nabla^2 f(\alpha \mathbf{x} + (1-\alpha)\mathbf{y})(\mathbf{y} - \mathbf{x}).$$

**Theorem 5** *Let* $f \in C^1$*. Then* $f$ *is convex over a convex set* $\Omega$ *if and only if*

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \nabla f(\mathbf{x})(\mathbf{y} - \mathbf{x})$$

*for all* $\mathbf{x}, \ \mathbf{y} \in \Omega$*.*

**Theorem 6** *Let* $f \in C^2$*. Then* $f$ *is convex over a convex set* $\Omega$ *if and only if the Hessian matrix of* $f$ *is positive semi-definite throughout* $\Omega$*.*

## System of Linear Equations

Solve for $\mathbf{x} \in \mathcal{R}^n$ from:

$$\begin{array}{rcl} \mathbf{a}_1\mathbf{x} & = & b_1 \\ \mathbf{a}_2\mathbf{x} & = & b_2 \\ \ldots & . & . \\ \mathbf{a}_m\mathbf{x} & = & b_m \end{array} \quad \Rightarrow A\mathbf{x} = \mathbf{b}$$

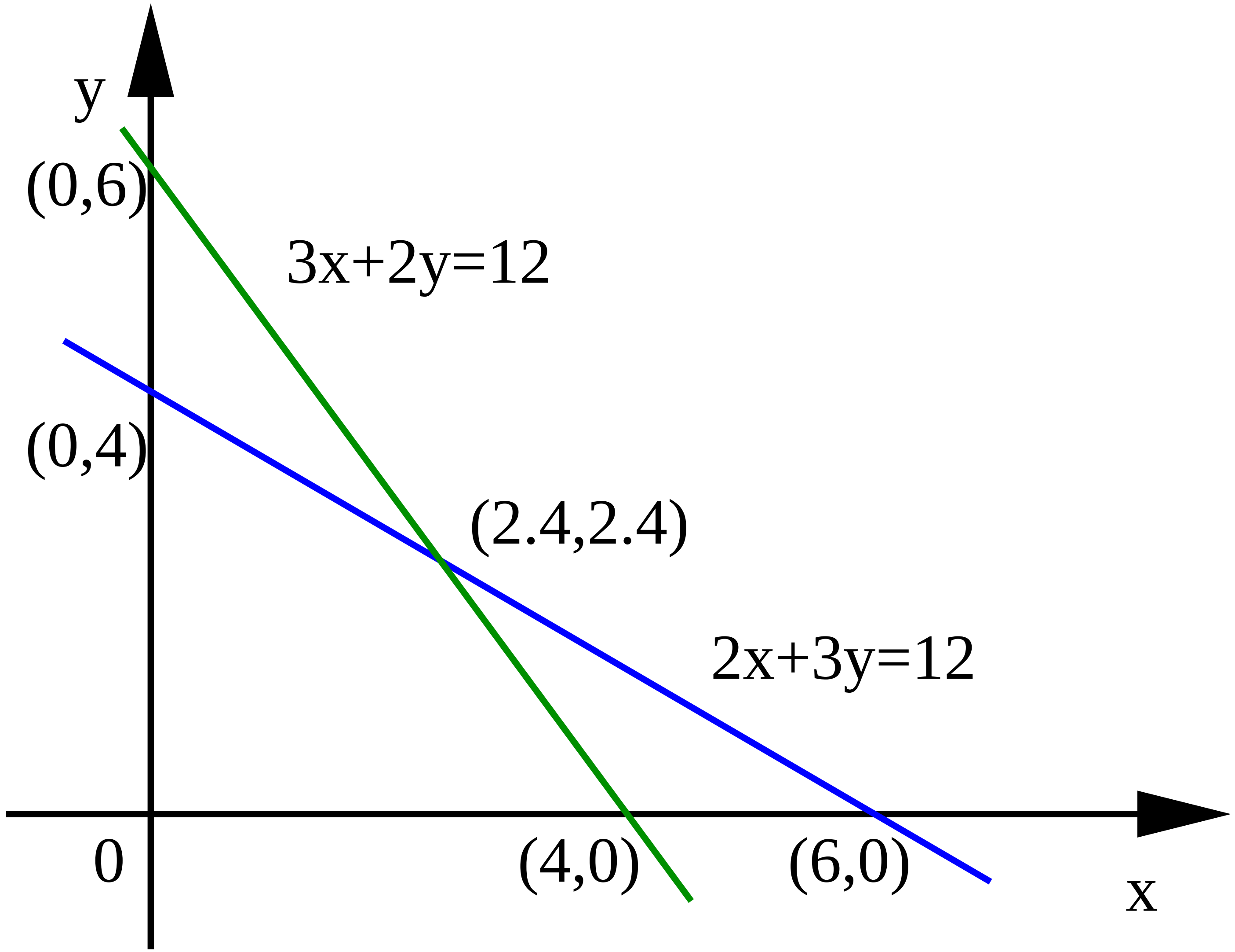


Figure 2: System of linear equations

## Fundamental Theorem of Linear Equations

**Theorem 7** *Given* $A \in \mathcal{R}^{m \times n}$ *and* $\mathbf{b} \in \mathcal{R}^m$*, the system* $\{\mathbf{x}: \ A\mathbf{x} = \mathbf{b}\}$ *has a solution if and only if that* $A^T\mathbf{y} = \mathbf{0}$ *and* $\mathbf{b}^T\mathbf{y} \neq 0$ *has no solution.*

A vector $\mathbf{y}$, with $A^T\mathbf{y} = \mathbf{0}$ and $\mathbf{b}^T\mathbf{y} \neq 0$, is called an infeasibility certificate for the system.

Example Let $A = (1; -1)$ and $\mathbf{b} = (1; 1)$. Then, $\mathbf{y} = (1/2; 1/2)$ is an infeasibility certificate.

Alternative systems: $\{\mathbf{x}: \ A\mathbf{x} = \mathbf{b}\}$ and $\{\mathbf{y}: \ A^T\mathbf{y} = \mathbf{0}, \ \mathbf{b}^T\mathbf{y} \neq 0\}$.

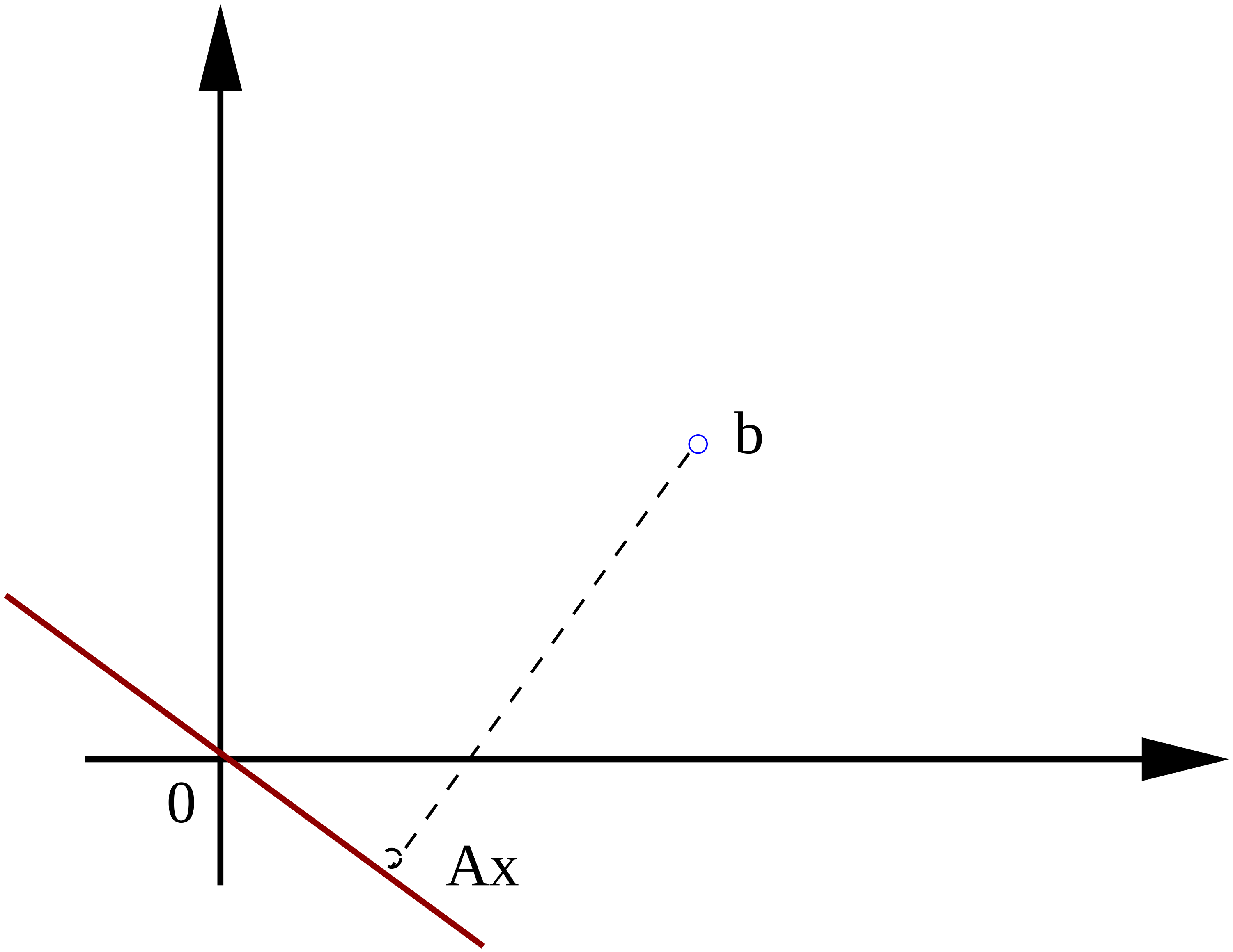


Figure 3: $\mathbf{b}$ is not in the set $\{A\mathbf{x} : \mathbf{x}\}$, and $\mathbf{y}$ is the distance vector from $\mathbf{b}$ to the set.

## Linear least-squares problem

Given $A \in \mathcal{R}^{m\times n}$ and $\mathbf{c} \in \mathcal{R}^n$,

$$\begin{array}{lll} (LS) & \text{minimize} & \|\mathbf{c} - A^T\mathbf{y}\|^2 \\ & \text{subject to} & \mathbf{y} \in \mathcal{R}^m. \end{array}$$

A close form solution:

$$AA^T\mathbf{y} = A\mathbf{c} \quad \text{or} \quad \mathbf{y} = (AA^T)^{-1}A\mathbf{c}.$$

$$\mathbf{c} - A^T\mathbf{y} = \mathbf{c} - A^T(AA^T)^{-1}A\mathbf{c} = \mathbf{c} - P\mathbf{c}$$

Projection matrix: $P = A^T(AA^T)^{-1}A$ or $P = I - A^T(AA^T)^{-1}A$.

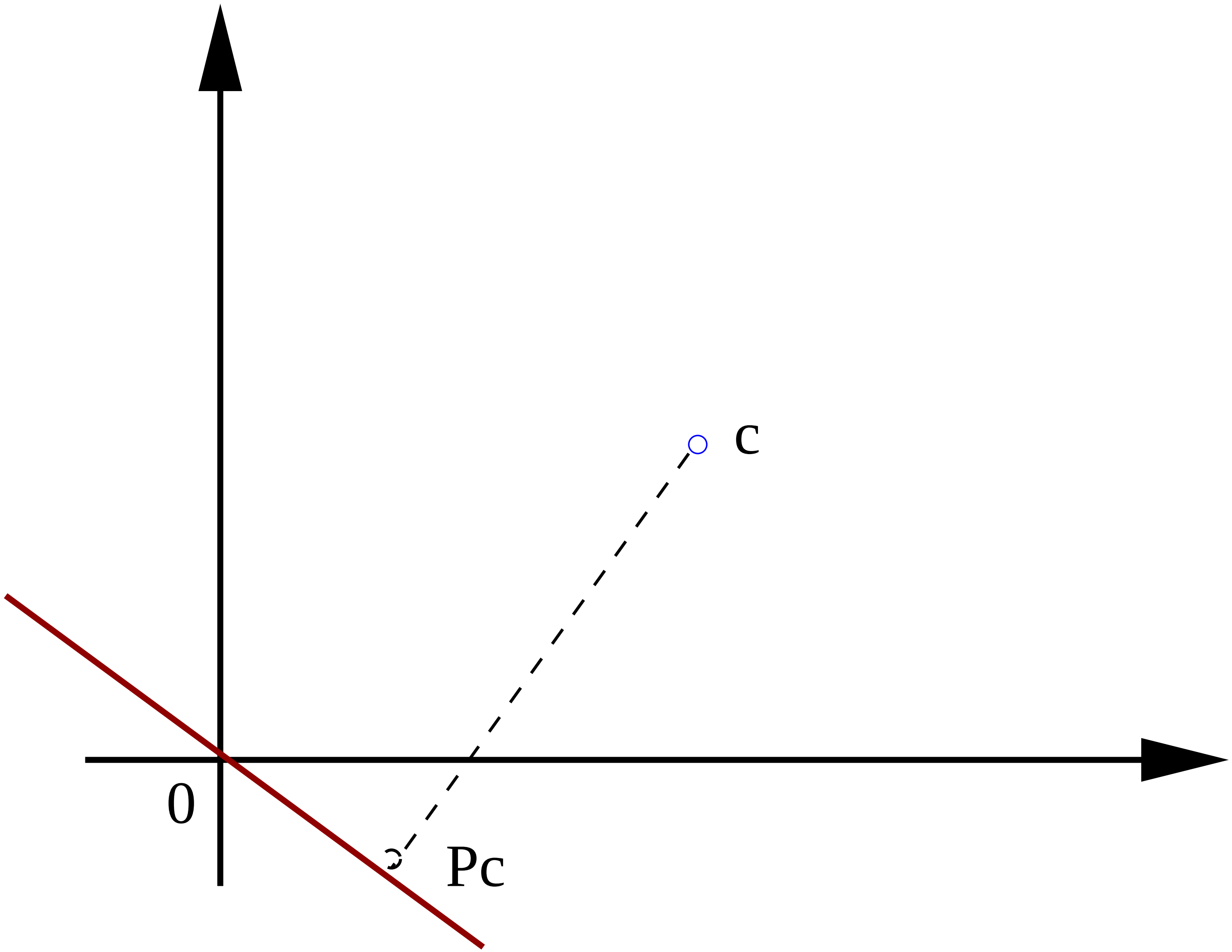


Figure 4: Projection of $\mathbf{c}$ onto a subspace

## Choleski decomposition method

$$AA^T = L\Lambda L^T$$

$$L\Lambda L^T \mathbf{y}^* = A\mathbf{c}$$

## System of nonlinear equations

Given $\mathbf{f}(\mathbf{x}) : \mathcal{R}^n \rightarrow \mathcal{R}^n$, the problem is to solve $n$ equations for $n$ unknowns:

$$\mathbf{f}(\mathbf{x}) = \mathbf{0}.$$

Given a point $\mathbf{x}^k$, Newton's Method sets

$$\mathbf{f}(\mathbf{x}) \simeq \mathbf{f}(\mathbf{x}^k) + \nabla \mathbf{f}(\mathbf{x}^k)(\mathbf{x} - \mathbf{x}^k) = \mathbf{0}.$$

$$\mathbf{x}^{k+1} = \mathbf{x}^k - (\nabla \mathbf{f}(\mathbf{x}^k))^{-1} \mathbf{f}(\mathbf{x}^k)$$

or solve for direction vector $\mathbf{d}_x$:

$$\nabla \mathbf{f}(\mathbf{x}^k)\mathbf{d}_x = -\mathbf{f}(\mathbf{x}^k) \quad \text{and} \quad \mathbf{x}^{k+1} = \mathbf{x}^k + \mathbf{d}_x.$$

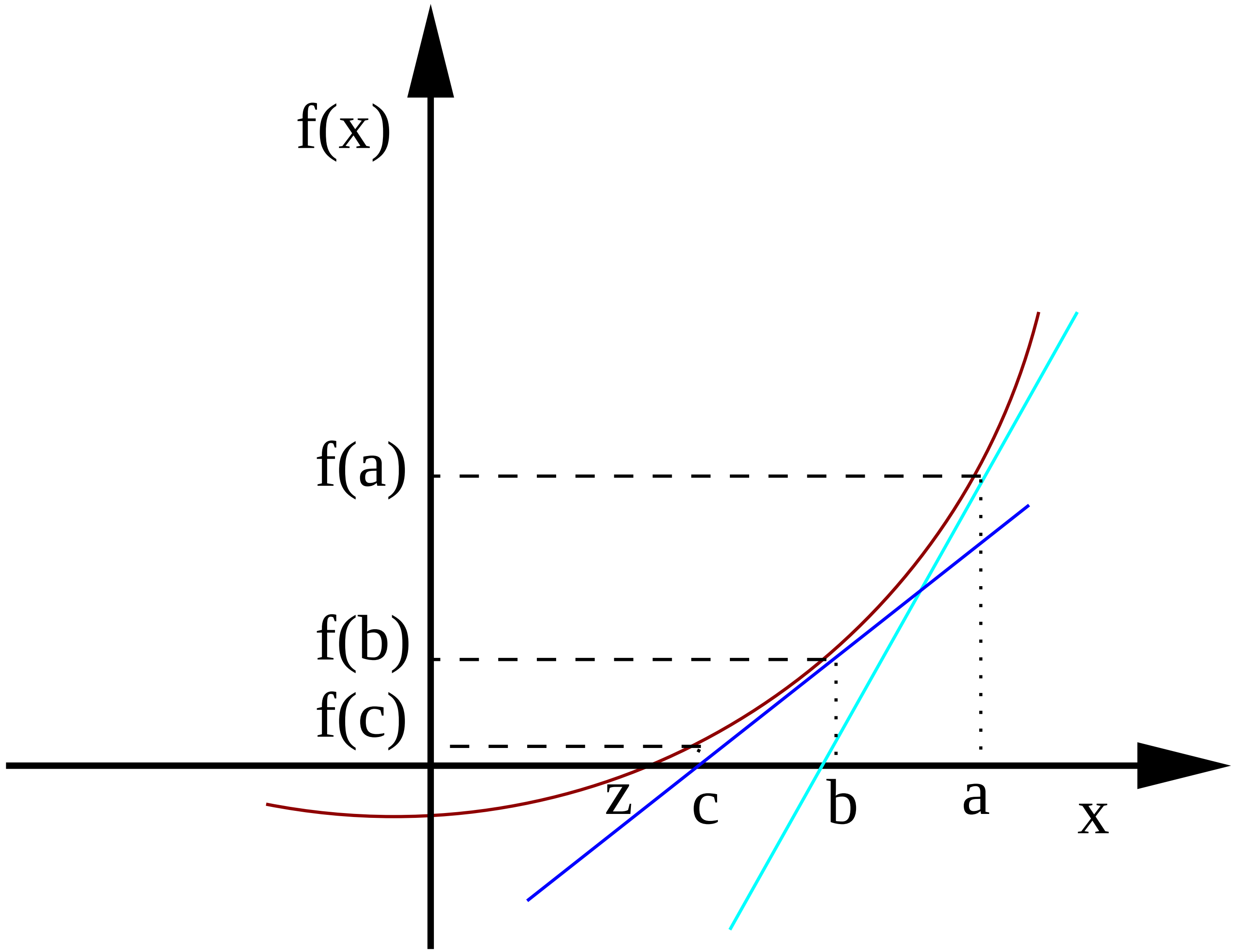


Figure 5: Newton's method for root finding

## The quasi Newton method

For minimization of objective function $f(\mathbf{x})$, then $\mathbf{f}(\mathbf{x}) = \nabla f(\mathbf{x})$

$$\mathbf{x}^{k+1} = \mathbf{x}^k - \alpha(\nabla^2 f(\mathbf{x}^k))^{-1}\nabla f(\mathbf{x}^k)$$

where scalar $\alpha \geq 0$ is called step-size. More generally

$$\mathbf{x}^{k+1} = \mathbf{x}^k - \alpha M^k \nabla f(\mathbf{x}^k)$$

where $M^k$ is an $n \times n$ symmetric matrix. In particular, if $M^k = I$, the method is called the gradient method, where $f$ is viewed as the gradient vector of a real function.

# Convergence and Big O

- $\{\mathbf{x}^k\}_0^\infty$ denotes a seqence $\mathbf{x}^0, \mathbf{x}^1, \mathbf{x}^2, ..., \mathbf{x}^k, ....$
- $\mathbf{x}^k \rightarrow \bar{\mathbf{x}}$ iff

$$\|\mathbf{x}^k - \bar{\mathbf{x}}\| \rightarrow 0$$

- $g(x) \geq 0$ is a real valued function of a real nonnegative variable, the notation $g(x) = O(x)$ means that $g(x) \leq \bar{c}x$ for some constant $\bar{c}$;
- $g(x) = \Omega(x)$ means that $g(x) \geq \underline{c}x$ for some constant $\underline{c}$;
- $g(x) = \theta(x)$ means that $\underline{c}x \leq g(x) \leq \bar{c}x$.
- $g(x) = o(x)$ means that $g(x)$ goes to zero faster than $x$ does:

$$\lim_{x \rightarrow 0} \frac{g(x)}{x} = 0$$