HOMEWORK ASSIGNMENT 1 SOLUTION

1.

x_i: number of shares of security i purchased.

1.

$$\begin{aligned}
\max \quad & s - 0.75x_1 - 0.35x_2 - 0.4x_3 - 0.95x_4 - 0.75x_5 \\
\text{s.t.} \quad & s - x_1 - x_3 - x_4 \le 0 \\
& s - x_1 - x_4 - x_5 \le 0 \\
& s - x_1 - x_3 - x_4 \le 0 \\
& s - x_2 - x_4 - x_5 \le 0 \\
& s - x_3 \le 0 \\
& x_1 \le 10, x_2 \le 5, x_3 \le 10 \\
& x_4 \le 10, x_5 \le 5 \\
& x_i \ge 0 \ \forall i \\
& s \quad \text{free}
\end{aligned}$$

2.

$$\begin{aligned}
\max \quad & s - 0.75x_1 - 0.35x_2 - 0.4x_3 - 0.95x_4 - 0.75x_5 \\
\text{s.t.} \quad & s - x_1 - x_3 - x_4 \le 0 \\
& s - x_1 - x_4 - x_5 \le 0 \\
& s - x_1 - x_3 - x_4 \le 0 \\
& s - x_2 - x_4 - x_5 \le 0 \\
& s - x_3 \le 0 \\
& x, s \quad \text{free}
\end{aligned}$$

2. Exercise 2.8-9.

Similar to the technique we use for absolute values, we can formulate the problem as follows.

$$\begin{aligned} \min \quad & z \\ \text{s.t.} \quad & c_i^T x + d_i \le z, 1 \le i \le p \\ & Ax = b \\ & x \ge 0. \end{aligned}$$

3. Exercise 2.8-10.

Define x_i to be the amount of units produced in the ith month; y_i to be the storage from month i to month $i+1$. And $z_i = (x_i - r)^+$. Then we can formulate the problem as follows:

$$\begin{aligned} \min \quad & \textstyle\sum_{i=1}^n \{sy_i + bx_i + (c-b)z_i\} & \\ \text{s.t.} \quad & x_i + y_{i-1} - d_i = y_i, & 1 \le i \le n, \\ & y_0 = y_n = 0, & \\ & z_i \ge x_i - r, & 1 \le i \le n, \\ & x_i, y_i, z_i \ge 0. & \end{aligned}$$

4. Exercise 2.8-13.

Consider the following system about variable z:

$$\begin{aligned} Az &= b \\ c^T z &= c^T x \\ z &\ge 0 \end{aligned}$$

Obviously, x is a feasible solution of this system. And Rank $\begin{pmatrix} A \\ c^T \end{pmatrix} \le m+1$. Thus we know the system must have a basic feasible solution. Assume it is y. Then the solution y satisfies the requirement.

5. Exercise 2.8-16.

Define $S(\in E^{n+m})$ to be the set of the feasible solutions to the second system. Then $S = \{(x,y) | Ax + y = b, x \ge 0, y \ge 0\}$. Define $S(\in E^n)$ to be the set of the feasible solutions to the first system, then $S^* = \{x | Ax \le b, x \ge 0\}$.

We consider the projection T from S to S^*: $T(x,y) = x$. It is linear. Noticing that for each $(x,y) \in S$, y is uniquely determined once x is fixed, it is not hard to prove T

is a one-to-one correspondence. Then we show that T is a one-to-one correspondence between extreme points of S and S^*.

For any extreme point $x \in S$, we prove $x^* = T(x)$ is also an extreme point in S^*. Otherwise, $x^* = ay^* + (1-a)z^*$, $y^*, z^* \in S^*$, $y^* \neq z^*$, $0 < a < 1$. Then, there must exist different $y, z \in S$, such that $x = ay + (1-a)z$. Therefore, x is not an extreme point, which contradicts with the assumption. So $T(x)$ must be an extreme point in S^*. Similarly, we can prove if x^* is an extreme point in S^*, then its corresponding point in S must be an extreme point too.

6.

(a) *Omitted.*

(b)

Extreme Point	*Defining Equations*
$O : (0, 0)$	$x_1 = x_2 = 0$
$A : (0, 2.5)$	$x_1 = x_3 = 0$
$B : (0.5, 3)$	$x_3 = x_6 = 0$
$C : (3, 3)$	$x_4 = x_6 = 0$
$D : (4, 2.5)$	$x_4 = x_5 = 0$
$E : (4, 0)$	$x_2 = x_5 = 0$

(c)

From	*to*	*Increasing*
O	A	x_2
A	B	x_1
B	C	x_3
C	D	x_6
D	E	x_4
E	O	x_5
O	E	x_1
E	D	x_2
D	C	x_5
C	B	x_4
B	A	x_6
A	O	x_3

7.

1. True: The set P lies in a set defined by $m = n - 1$ linearly independent con-

straints, that is, of an one–dimensional affine subspace. Hence, every solution x of $Ax = b$ is of the form $x = x^0 + \lambda x^1$ for some scalar λ, where x^0 is a point in P and x^1 is some nonzero directional vector. Thus, P is constrained to lie on a line and cannot have more than two extreme points.

2. False: Consider min x_1 s.t. $x_1 = 1$, $(x_1; x_2) \geq 0$. The optimal solution set is unbounded.

3. False: Consider a linear program where $c = 0$. Then any feasible solution is optimal, no matter how many positive components it has.

4. True: If x and y are optimal, then so is any convex combination of them; thus, the optimal solution set is convex.

5. False: Consider min x_1 s.t. $x_1 = 0$, $(x_1; x_2) \geq 0$. The optimal solution set is $\{(0; x_2) : x_2 \geq 0\}$. It has only one optimal basic feasible solution.

6. False: Consider min $|x_1 - 0.5|$ s.t. $x_1 + x_2 = 1$, $(x_1; x_2) \geq 0$. Note that:

$$|x_1 - 0.5| = \max\{x_1 - 0.5, 0.5 - x_1\}$$

The optimal solution is $(0.5; 0.5)$ and it is unique; and it is not an extreme point.

8.

1. For any $c_1, c_2 \in R^n$, consider

$$\begin{aligned} Z_1 := \min\ & (\lambda c_1 + (1-\lambda)c_2)^T x \\ s.t.\ & Ax = b \\ & x \in K \end{aligned} \tag{1}$$

$$\begin{aligned} Z_2 := \min\ & \lambda c_1^T x_1 + (1-\lambda) c_2^T x_2 \\ s.t.\ & Ax_1 = b \\ & Ax_2 = b \\ & x_1, x_2 \in K \end{aligned} \tag{2}$$

Since every feasible solution to (1) is also feasible to (2) and has the same objective function value, $Z_1 \geq Z_2$. Hence,

$$z(\lambda c_1 + (1-\lambda)c_2) = Z_1 \geq Z_2 = \lambda z(c_1) + (1-\lambda) z(c_2).$$

2. $\mathbf{c} = (c_1, c_2, \cdots, c_n)^T$. WLOG, assume $c_1 \le c_2 \le \cdots \le c_n$. Then the optimal solution $x^* = \sum_{i=1}^k e_i$, where $e_i \in R^n$ is the unit vector in ith coordinate. Otherwise, $\exists i \le k, j > k$, such that $x_i < 1$, $x_j > 0$. By replacing $x_i' = 1$, $x_j' = x_i + x_j - 1$, the value of the objective function decreases, a contradiction!

3. Let $t^1, t^2 > 0$. For any given $0 \le \alpha \le 1$, we have

$$\begin{aligned} f\left(\frac{\alpha x^1 + (1-\alpha)x^2}{\alpha t^1 + (1-\alpha)t^2}\right) &= f\left(\frac{\alpha t^1 \cdot (x^1/t^1) + (1-\alpha)t^2 \cdot (x^2/t^2)}{\alpha t^1 + (1-\alpha)t^2}\right) \\ &= f\left(\frac{\alpha t^1}{\alpha t^1 + (1-\alpha)t^2} \cdot \frac{x^1}{t^1} + \frac{(1-\alpha)t^2}{\alpha t^1 + (1-\alpha)t^2} \cdot \frac{x^2}{t^2}\right) \\ &\le \frac{\alpha t^1}{\alpha t^1 + (1-\alpha)t^2} \cdot f(x^1/t^1) + \frac{(1-\alpha)t^2}{\alpha t^1 + (1-\alpha)t^2} \cdot f(x^2/t^2). \end{aligned}$$

Here, we used the convexity of f and

$$\frac{\alpha t^1}{\alpha t^1 + (1-\alpha)t^2} + \frac{(1-\alpha)t^2}{\alpha t^1 + (1-\alpha)t^2} = 1$$

and each of them is non-negative.

Thus,

$$(\alpha t^1 + (1-\alpha)t^2) \cdot f\left(\frac{\alpha x^1 + (1-\alpha)x^2}{\alpha t^1 + (1-\alpha)t^2}\right) \le \alpha t^1 \cdot f(x^1/t^1) + (1-\alpha)t^2 \cdot f(x^2/t^2),$$

that is, $t \cdot f(x/t)$ is convex by definition.

9. We show that the optimal solution of $\mathbf{y}^* = (\mathbf{y_1^*}, ..., \mathbf{y_m^*})$ satisfy

$$y_i^* = \min_{j \in \mathcal{A}_i} \{c_j + \gamma \mathbf{p_j} \mathbf{y}^*\}$$

for all i. If there exists i such that

$$y_i^* < \min_{j \in \mathcal{A}_i} \{c_j + \gamma \mathbf{p_j} \mathbf{y}^*\}.$$

Then, let

$$y_i' = \min_{j \in \mathcal{A}_i} \{c_j + \gamma \mathbf{p_j} \mathbf{y}^*\}$$

and $y_k' = y_k^*$ for all $k \neq i$. Then, $\mathbf{y}' = (y_1', ..., y_m')$ satisfies the constraints but achieves a larger objective value. This contradicts the optimality of $\mathbf{y}^*$.