Question 1.

This problem is concerned with using an optimality criterion for linear programming to decide whether the vector $x^* = (0, 2, 0, 7, 0)$ is optimal for the linear program

$$
(\mathrm{P})\qquad
\begin{array}{lrcrcrcrcrcr}
\text{maximize} & 8x_1 & - & 9x_2 & + & 12x_3 & + & 4x_4 & + & 11x_5 & & \\
\text{subject to} & 2x_1 & - & 3x_2 & + & 4x_3 & + & x_4 & + & 3x_5 & \le & 1 \\
 & x_1 & + & 7x_2 & + & 3x_3 & - & 2x_4 & + & x_5 & \le & 1 \\
 & 5x_1 & + & 4x_2 & - & 6x_3 & + & 2x_4 & + & 3x_5 & \le & 22 \\
 & & & & & & & & & x_1, x_2, x_3, x_4, x_5 & \ge & 0
\end{array}
$$

1. Write down the dual (D) of the problem (P).
2. Noting that the constraints of (P) have the form $Ax \le b,\ x \ge 0$, write down the basic variables at solution x^* in the canonical form.
3. For the moment, assume that x^* *is* an optimal solution of (P). Write down a system of equations in the *dual variables* that must hold as a consequence of this assumption.
4. Solve for the dual solution according to the system in part (3) and verify whether the vector x^* is optimal for the linear program.

1. Write down the dual (D) of the problem (P).

   The dual problem (D) is

$$
\begin{array}{lrcrcrcr}
\text{minimize} & y_1 & + & y_2 & + & 22y_3 & & \\
\text{subject to} & 2y_1 & + & y_2 & + & 5y_3 & \ge & 8 \\
 & -3y_1 & + & 7y_2 & + & 4y_3 & \ge & -9 \\
 & 4y_1 & + & 3y_2 & - & 6y_3 & \ge & 12 \\
 & y_1 & - & 2y_2 & + & 2y_3 & \ge & 4 \\
 & 3y_1 & + & y_2 & + & 3y_3 & \ge & 11 \\
 & & & & & y_1, y_2, y_3 & \ge & 0
\end{array}
$$

2. Noting that the constraints of (P) have the form $Ax \le b,\ x \ge 0$, write down the basic variables at solution x^* in the canonical form.

   Let the slack variables for the three constraints be x_6, x_7, and x_8 from top down. Then

   Basic varaibles are $\underline{\quad \{2, 4, 7\} \quad}$

3. For the moment, assume that x^* *is* an optimal solution of (P). Write down a system of equations in the *dual variables* that must hold as a consequence of this assumption.

   From part (2), we find that the second and fourth of the dual constraints must hold as equations. Likewise, the second dual variable must be zero. Thus, we have

$$\begin{aligned} -3y_1 + 7y_2 + 4y_3 &= -9 \\ y_1 - 2y_2 + 2y_3 &= 4 \\ y_2 \quad\quad &= 0 \end{aligned}$$

4. Solve for the dual solution according to the system in part (3) and verify whether the vector x^* is optimal for the linear program.

   After substituting $y_2 = 0$ into the first two equations, we obtain a system of two equations in two unknowns. The solution of the latter system is

$$(y_1,\, y_3) = \left(\frac{17}{5},\, \frac{3}{10}\right).$$

   So, the assumption that x^* is optimal leads to the conclusion that the values of the dual variables must be

$$(y_1,\, y_2,\, y_3) = \left(\frac{17}{5},\, 0,\, \frac{3}{10}\right).$$

   It is tempting to conclude that x^* is optimal because the primal and dual objective functions have the same value at

$$x^* = (0, 2, 0, 7, 0) \text{ and } (y_1,\, y_2,\, y_3) = \left(\frac{17}{5},\, 0,\, \frac{3}{10}\right),$$

   respectively, namely 10. This would be the correct answer if $y = \left(\frac{17}{5},\, 0,\, \frac{3}{10}\right)$ were feasible in (D). But it is not feasible for (D)—the third dual constraint is violated. Hence the correct answer is that x^* is not optimal for (P).

∎

Question 2.

Consider the problem

$$(P) \quad \begin{array}{ll} \max & -p_1 x_1 + p_2 x_2 \\ \text{s.t.} & x_1 - x_2 = 0 \\ & 0 \le x_j \le 1,\ j = 1, 2 \end{array}$$

Discuss the range of optimal feasible solutions to (P) and its dual (D) in the three cases:

(a) $p_1 < p_2$

(b) $p_1 = p_2$

(c) $p_1 > p_2$

The primal and the dual are

$$(P) \quad \begin{array}{ll} \max & -p_1x_1 + p_2x_2 \\ \text{s.t.} & x_1 - x_2 = 0 \\ & 0 \le x_j \le 1, \;\; j = 1, 2 \end{array} \qquad (P) \quad \begin{array}{ll} \min & y_2 + y_3 \\ \text{s.t.} & y_1 + y_2 \ge -p_1 \\ & y_1 - y_3 \le -p_2 \\ & 0 \le y_j, \;\; j = 2, 3 \end{array}$$

(a) $p_1 < p_2$. In this case, the optimal solution for the primal is $(x_1, x_2) = (1, 1)$. The optimal dual soution is $y_2 = t, y_1 = -p_1 - t, y_3 = p_2 - p_1 - t$ for any $t \in [0, p_2 - p_1]$.

(b) $p_1 = p_2$. In this case, any feasible solution is an optimal solution for the primal. The optimal dual soution is $y_2 = y_3 = 0, y_1 = -p_1 = -p_2$.

(c) $p_1 > p_2$. In this case, the optimal solution for the primal is $(x_1, x_2) = (0, 0)$. The optimal dual soution is $y_2 = y_3 = 0$ and any $y_1 \in [-p_1, -p_2]$. ∎

Question 3. Consider the linear inequality system

$$\mathcal{P} = \{y \in \Re^m : \;\; A^T y \le c\}$$

where $A \in \Re^{m \times n}$ with rank m and $m < n$, and $c \in \Re^n$, and the linear program

$$\min \; c^T x \quad \text{subject to} \quad x \in \mathcal{X} = \{x \in \Re^n : \;\; Ax = 0, \;\; x \ge 0\}$$

Show

i $\mathcal{P}$ is feasible if and only if the minimal value of the linear program is 0.

ii $\mathcal{P}$ has a nonempty interior if and only if the linear program has the unique all 0 minimal solution.

iii $\mathcal{P}$ is bounded if and only if $\mathcal{X}$ has a nonempty interior, that is, there is $x > 0$ and $Ax = 0$.

Proof i This is identical to Farkas' lemma.

Proof ii Let x^* be any optimal solution and $\bar{s}$ be an interior slack solution, that is, $\bar{s} > 0$. Then, $c^T x^* = 0$ and $Ax^* = 0$ imply that $\bar{s}^T x^* = 0$ which further implies that $x^* = 0$. The reverse is true due to the strictly complementarity partition theorem.

Proof iii Let $\bar{x} > 0$ and $A\bar{x} = 0$, then, for any point in $\mathcal{P}$ we have

$$c^T \bar{x} = (c - A^T y)^T \bar{x} = s^T \bar{x}.$$

Thus, for every j, $s_j \bar{x}_j \le c^T \bar{x}$ or $s_j \le (c^T \bar{x}) / \bar{x}_j$. This implies that the dual slack variables are bounded or $\mathcal{P}$ is bounded.

On the other hand, if at least one variable in x has to be zero at every feasible solution to $x \geq 0$ and $Ax = 0$, then, from the strictly complementarity partition theorem, there is $\bar{s} \neq 0$, $\bar{s} \geq 0$ and $\bar{s} = -A^T\bar{y}$. Let y be any feasible point in $\mathcal{P}$, then $y(\alpha) = y + \alpha \cdot \bar{y}$ is also in $\mathcal{P}$ for any $\alpha \geq 0$, which imply that $\mathcal{P}$ is unbounded. ∎

Question 4. Prove Lemmas 1 in Lecture Note #10. They are: Given $(x > 0, y, s > 0)$ interior feasible solution for the standard form linear program, let the direction $d = (d_x, d_y, d_s)$ be generated by equation

$$\begin{aligned} Sd_x + Xd_s &= \gamma\mu e - Xs, \\ Ad_x &= 0, \\ -A^Td_y - d_s &= 0 \end{aligned}$$

with $\gamma = n/(n+\rho)$, and let

$$\theta = \frac{\alpha\sqrt{\min(Xs)}}{\|(XS)^{-1/2}(\frac{x^Ts}{(n+\rho)}e - Xs)\|} \ , \tag{1}$$

where α is a positive constant less than 1. Let

$$x^+ = x + \theta d_x, \quad y^+ = y + \theta d_y, \quad \text{and} \quad s^+ = s + \theta d_s.$$

Then, we have (x^+, y^+, s^+) remains interior feasible and

$$\begin{aligned} &\psi_{n+\rho}(x^+, s^+) - \psi_{n+\rho}(x, s) \\ \leq\ & -\alpha\sqrt{\min(Xs)}\|(XS)^{-1/2}(e - \frac{(n+\rho)}{x^Ts}Xs)\| + \frac{\alpha^2}{2(1-\alpha)}, \end{aligned}$$

where

$$\psi_{n+\rho}(x, s) = (n+\rho)\log(s^Tx) - \sum_{j=1}^{n}\log(s_jx_j).$$

Proof of Lemma 1. Note that $d_x^Td_s = 0$ by the facts that $Ad_x = 0$ and $d_s = -A^Td_y$. Recall that

$$\psi_{n+\rho}(x, s) = (n+\rho)\log(s^Tx) - \sum_{j=1}^{n}\log(s_jx_j).$$

and

$$x^+ = x + \theta d_x, \quad y^+ = y + \theta d_y, \quad \text{and} \quad s^+ = s + \theta d_s.$$

Then it is clear that

$$\begin{aligned} &\psi_{n+\rho}(x^+, s^+) - \psi_{n+\rho}(x, s) \\ =\ & (n+\rho)\log\left(1 + \frac{\theta d_s^Tx + \theta d_x^Ts}{s^Tx}\right) - \sum_{j=1}^{n}\left(\log(1 + \frac{\theta d_{s_j}}{s_j}) + \log(1 + \frac{\theta d_{x_j}}{x_j})\right) \\ \leq\ & (n+\rho)\theta\frac{d_s^Tx + d_x^Ts}{s^Tx} - \theta e^T(S^{-1}d_s + X^{-1}d_x) + \frac{||\theta S^{-1}d_s||^2 + ||\theta X^{-1}d_x||^2}{2(1-\tau)}. \end{aligned}$$

where $\tau = \max\{||\theta S^{-1}d_s||_\infty, ||\theta X^{-1}d_x||_\infty\}$, and the last inequality follows from Karmakar's Lemma. In the following, we first bound the difference of the first two terms and then bound the third term.

For simplicity, we let $\beta = \frac{n+\rho}{s^T x}$. Then we have

$$\gamma\mu e = \frac{n}{n+\rho}\frac{s^T x}{n}e = \frac{1}{\beta}e.$$

The difference of the first two terms

$$\begin{aligned}
& \beta\theta(d_s^T x + d_x^T s) - \theta e^T(S^{-1}d_s + X^{-1}d_x) \\
= & \beta\theta e^T(Xd_s + Sd_x) - \theta e^T(S^{-1}d_s + X^{-1}d_x) \\
= & \theta(\beta e - (XS)^{-1}e)^T(Xd_s + Sd_x) \\
= & \theta(\beta e - (XS)^{-1}e)^T(\frac{1}{\beta}e - Xs) \quad \text{(By Newton's equations)} \\
= & -\theta(e - \beta Xs)^T(XS)^{-1}(\frac{1}{\beta}e - Xs) \\
= & -\theta\beta(\frac{1}{e} - Xs)^T(XS)^{-1}(\frac{1}{\beta}e - Xs) \\
= & -\theta\beta||(XS)^{-1/2}(\frac{1}{\beta}e - Xs)||^2 \\
= & -\beta\alpha\sqrt{\min(Xs)}||(XS)^{-1/2}(\frac{1}{\beta}e - Xs)|| \quad \text{(By the definition of } \theta) \\
= & -\alpha\sqrt{\min(Xs)}||(XS)^{-1/2}(e - \beta Xs)|| \\
= & -\alpha\sqrt{\min(Xs)}||(XS)^{-1/2}(e - \frac{n+\rho}{x^T s}Xs)||.
\end{aligned} \tag{2}$$

Now we will try to bound

$$\frac{||\theta S^{-1}d_s||^2 + ||\theta X^{-1}d_x||^2}{2(1-\tau)}.$$

Note that

$$S^{-1}d_s = (XS)^{-1/2}(XS)^{-1/2}Xd_s, \quad \text{and} \quad X^{-1}d_x = (XS)^{-1/2}(XS)^{-1/2}Sd_x.$$

Therefore,

$$\begin{aligned}
& ||\theta S^{-1}d_s||^2 + ||\theta X^{-1}d_x||^2 \\
\leq & \theta^2\frac{||(XS)^{-1/2}Xd_s||^2 + ||(XS)^{-1/2}Sd_x||^2}{\min(Xs)} \\
\leq & \theta^2\frac{||(XS)^{-1/2}Xd_s + (XS)^{-1/2}Sd_x||^2}{\min(Xs)}
\end{aligned}$$

$$
\begin{aligned}
&= \theta^2 \frac{||(XS)^{-1/2}(Xd_s + Sd_x)||^2}{\min(Xs)} \\
&= \theta^2 \frac{||(XS)^{-1/2}(\frac{1}{\beta}e - Xs)||^2}{\min(Xs)} \\
&= \alpha^2
\end{aligned}
$$

where the first equality holds since

$$d_s^T X^T (XS)^{-1/2}(XS)^{-1/2} S d_x = d_s^T d_x = 0$$

and the last equality follows from the definition of θ. This also show that $\tau \leq \alpha$. Therefore, we must have

$$\frac{||\theta S^{-1}d_s||^2 + ||\theta X^{-1}d_x||^2}{2(1-\tau)} \leq \frac{\alpha^2}{2(1-\alpha)}. \tag{3}$$

The desired inequality follows from (2) and (3).

To complete the proof of Lemma 1, we should show that

$$x^+ > 0, \quad \text{and} s^+ > 0$$

$$Ax^+ = b$$

$$c - A^T y^+ = s^+.$$

The last two equalities are easily verified by the definition of d_x, d_y, d_s. To prove the inequalities, we recall that

$$||\theta S^{-1}d_s||^2 + ||\theta X^{-1}d_x||^2 \leq \alpha^2 < 1$$

and thus $\tau < 1$. ∎