Resolution of degeneracy using Bland's rule

The pivot selection rule of R.G. Bland is easily stated.[1] It is a *double least-index rule* consisting of the following two parts:

(i) Among all candidates for the entering column (i.e., those with $\bar{c}_j < 0$), choose the one with the smallest index, say s.

(ii) Among all rows i for which the minimum ratio test results in a tie, choose the row r for which the corresponding basic variable has the smallest index, j_r.

Note that in (ii), the number r itself need not be smallest row number among all those rows involved in a tie. It is the index of the associated basic variable that is the smallest among all such indices. Thus, with

$$s = \arg\min\{j : \bar{c}_j < 0\}$$

we define r by the condition

$$j_r = \min\left\{j_i : i \in \arg\min\left\{\frac{\bar{b}_i}{\bar{a}_{is}} : \bar{a}_{is} > 0\right\}\right\}.$$

Theorem. Under Bland's Pivot Selection Rule, the Simplex Algorithm cannot cycle.

Proof. The argument is complicated, yet very elementary. For the most part, the development given below follows the one used by Bland. Let x_0 denote $-z$. We think of the initial data as being expressed in a tableau of $m+1$ rows and $n+2$ columns which we write as

$$\mathcal{A} = \begin{bmatrix} 1 & c^{\mathrm{T}} & 0 \\ 0 & A & b \end{bmatrix}.$$

One row (the 0^{th}, $\mathcal{A}_{0\bullet}$) and column (the 0^{th}, $\mathcal{A}_{\bullet 0}$) pertain the variable x_0 we wish to optimize. The column indexed by $n+1$ is the right-hand side of the system of equations (augmented by an equation for the objective function).

Let $\bar{\mathcal{A}}$ denote the first $n+1$ columns of $\mathcal{A}$, i.e., with column $\mathcal{A}_{\bullet n+1}$ deleted. Analogous notations will be used for pivotal transforms of $\bar{\mathcal{A}}$.

[1] Bland's rule relies on the existence of a linear ordering of the indices used. When the indices are numerical, for example $i \in \{1,\ldots,m\}$ and $j \in \{1,\ldots,n\}$, this is not a problem. When the indices are alphanumerical, then a lexicographic (dictionary) ordering can be used.

Now if cycling occurs, there is a set τ of indices $j \in \{1, \ldots, n\}$ such that x_j becomes basic during cycling. Clearly τ has only a finite number of elements, so it has a largest element which we denote by q. Let $\mathcal{A}'$ denote the tableau that first specifies q as the pivot column. This means that x_q becomes a basic variable in the next tableau.

Let $y = (y_0, y_1, \ldots, y_n) = \bar{\mathcal{A}}'_{0\bullet}$. By virtue of the definition of q and the rule that results in the choice of q, we have

$$y_0 = 1, \quad y_j \geq 0 \quad 1 \leq j < q, \quad y_q < 0. \tag{1}$$

Note that the $(n+1)$-vector y belongs to the row space of $\bar{\mathcal{A}}$. Now x_q must also leave the basis at some tableau $\mathcal{A}''$. Let $x_q = x_{j_r}$, and let t denote the pivot column when x_q becomes nonbasic. Define the $(n+1)$-vector $v = (v_0, v_1, \ldots, v_n)$ as follows:

$$v_{j_i} = \bar{a}''_{it} \quad i = 0, 1, \ldots, m, \quad v_t = -1, \quad v_j = 0 \quad \text{else}. \tag{2}$$

Note that $v_0 = v_{j_0} = \bar{a}''_{0t} < 0$, $v_q = \bar{a}''_{rt} > 0$, and v is in the null space of $\bar{\mathcal{A}}$. Thus, $y \cdot v = 0$, and by construction $y_0 v_0 < 0$. Hence $y_j v_j > 0$ for some $j \geq 1$. Since $y_j \neq 0$, x_j must be nonbasic in $\mathcal{A}'$; since $v_j \neq 0$, then either x_j is basic in $\mathcal{A}''$ or else $j = t$. Accordingly, $j \in \tau$, and hence $j \leq q$. By the construction again, $y_q < 0 < v_q$ which implies that $y_q v_q < 0$ hence $j < q$.

Furthermore, (1) implies that $y_j > 0$, so $v_j > 0$. Next we observe that $v_t = -1$ implies $j \neq t$. All these lead to the conclusion that x_j is currently basic in $\mathcal{A}''$. Let $j = j_p$ for some p. Then $v_j = \bar{a}''_{pt} > 0$.

Note that during the cycling the right-hand-side vector $\bar{b}$ does not change and the values of all variables in τ are fixed at 0. This implies $\bar{b}''_p = 0$. We have established that $j = j_p$, $\bar{a}''_{pt} > 0$ and $\bar{b}''_p = 0$. But this contradicts the assumption that x_q is removed from the basic set corresponding to tableau $\mathcal{A}''$, since $j < q$ and by Bland's rule j should be removed. This means that cycling cannot occur when Bland's Rule is applied. ■

Remark. This elegant degeneracy resolution rule has the drawback that it may result in pivot choices that do not *significantly* improve the objective function value. It may also force the selection of dangerously small pivot elements.