Project Report
MS&E 310
Online Linear Programming

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1 Question 1

The problem can be rewritten as follows:

$$\begin{array}{lllr}
\min & -\{(\sum_{j=1}^{n}\pi_j x_j)+u(s)\} & & \\
\text{s.t.} & \sum_{j=1}^{n} a_{ij}x_j + s_i = b_i & i=1,2,\cdots,m & \lambda\ free \\
 & x_j - 1 \leq 0 & j=1,2,\cdots,n & \gamma \leq 0 \\
 & x_j \geq 0 & j=1,2,\cdots,n & \alpha \geq 0 \\
 & s_i \geq 0 & i=1,2,\cdots,m & \beta \geq 0.
\end{array} \tag{1}$$

As such, the Lagrangian function is:

$$\begin{aligned}
L(x,s,\gamma,\alpha,\beta) = & -(\sum_{j=1}^{n}\pi_j x_j) - u(s) + \sum_{i=1}^{m}\lambda_i(\sum_{j=1}^{n}a_{ij}x_j + s_i - b_i) \\
& - \sum_{j=1}^{n}\gamma_j(x_j-1) - \sum_{j=1}^{n}\alpha_j x_j - \sum_{i=1}^{m}\beta_j s_j
\end{aligned} \tag{2}$$

Setting $\nabla_{x,s}L(x,s,\lambda,\gamma,\alpha,\beta)=0$ results in the conditions below:

$$\begin{array}{ll}
\pi - A^T\lambda + \gamma + \alpha = 0 & \\
\nabla u(s) - \lambda + \beta = 0 & \\
\lambda_i(\sum_{j=1}^{n}a_{ij}x_j + s_i - b_i) & i=1,2,\cdots,m \\
\gamma_j(x_j-1) & j=1,2,\cdots,n \\
\alpha_j x_j & j=1,2,\cdots,n \\
\beta_j s_j & i=1,2,\cdots,m.
\end{array} \tag{3}$$

Eliminating α,β from the above conditions results in the following set of KKT coinditions:

$$\begin{array}{ll}
\pi - A^T\lambda + \gamma \leq 0 & \\
x_j(\pi_j - \sum_{i=1}^{m}\lambda_i a_{ij} + \gamma_j) & j=1,2,\cdots,n \\
\nabla u(s) - \lambda \leq 0 & \\
s_i(\frac{\partial u(s)}{\partial s_i} - \lambda_i) = 0 & i=1,2,\cdots,m \\
\gamma_j(x_j-1) & j=1,2,\cdots,n \\
\lambda\ free, \gamma \leq 0, x \geq 0, s \geq 0 &
\end{array} \tag{4}$$

The objective function is strictly concave. In addition, the feasibe region is convex. As a result, the KKT conditions are sufficient for global optimality.

Proof the pricing is unique

Lemma 1. *If (x_1, s_1) and (x_2, s_2) are two optimal solutions to (1), then $s_1 = s_2$.*

Proof. Towards a contradiction, suppose $s_1 \neq s_2$. Define $f(x, s) = -\pi x - u(s)$ to be the objective function. Hence, $f(x_1, s_1) = f(x_2, s_2)$.

Moroever, $f(x, s)$ is convex. So, for $0 \leq \alpha \leq 1$:

$$\begin{aligned} &\alpha f(x_1, s_1) + (1-\alpha) f(x_2, s_2) \geq f(\alpha x_1 + (1-\alpha)x_2, \alpha s_1 + (1-\alpha)s_2) \\ \Rightarrow &f(x_1, s_1) \geq f(\alpha x_1 + (1-\alpha)x_2, \alpha s_1 + (1-\alpha)s_2) \\ \Rightarrow &f(x_1, s_1) = f(\alpha x_1 + (1-\alpha)x_2, \alpha s_1 + (1-\alpha)s_2) \end{aligned} \tag{5}$$

On the other hand, $u(\cdot)$ is strictly concave. Therefore,

$$\alpha u(s_1) + (1-\alpha)u(s_2) < u(\alpha s_1 + (1-\alpha)s_2)$$

In addition,

$$\alpha \pi x_1 + (1-\alpha)\pi x_2 = \pi(\alpha x_1 + (1-\alpha)x_2)$$

Therefore,

$$\begin{aligned} &-\alpha(\pi x_1 + u(s_1)) - (1-\alpha)(\pi x_2 + u(s_2)) > -\pi(\alpha x_1 + (1-\alpha)x_2) - u(\alpha s_1 + (1-\alpha)s_2) \\ \Rightarrow \quad &\alpha f(x_1, s_1) + (1-\alpha) f(x_2, s_2) > f(\alpha x_1 + (1-\alpha)x_2, \alpha s_1 + (1-\alpha)s_2) \\ \Rightarrow \quad &f(x_1, s_1) > f(\alpha x_1 + (1-\alpha)x_2, \alpha s_1 + (1-\alpha)s_2) \end{aligned} \tag{6}$$

But (5) and (6) contradict.

□

Lemma 1 implies for the optimal solution (x^*, s^*), s^* is unique.

On the other hand, the assumption $\frac{\partial u(\cdot)}{\partial s_i}|_{s_i=0} = \infty$ implies: $s_i > 0$ for all i. Therefore, from the KKT conditions: $\lambda = \nabla u(s)$. As a result, λ is unique.

Interpretation of u(s) and s

- s: Presents the amount of resources left for future bids.
- u(s): Presents a penalty function that ensures the market maker finds an allocation that is pari-mutual and also makes the pricing (λ) unique.

2 Question 2

For each k, the problem can be rewritten as follows:

$$
\begin{array}{llll}
\min & -\pi_k x_k - u(s) & & \\
\text{s.t.} & a_{ik}x_k + s_i = b_i - q_i^{k-1} & i = 1, 2, \cdots, m & \lambda \ free \\
& x_k - 1 \le 0 & & \gamma_k \le 0 \\
& x_k \ge 0 & & \alpha_k \ge 0 \\
& s_i \ge 0 & i = 1, 2, \cdots, m & \beta \ge 0.
\end{array} \tag{7}
$$

As such, the Lagrangian function is:

$$
\begin{aligned}
L(x_k, s, \lambda, \gamma_k, \alpha_k, \beta) = & -\pi_k x_k - u(s) + \sum_{i=1}^{m} \lambda_i (a_{ik}x_k + s_i - b_i + q_i^{k-1}) \\
& -\gamma_k(x_k - 1) - \alpha_k x_k - \sum_{i=1}^{m} \beta_j s_j
\end{aligned} \tag{8}
$$

Setting $\nabla_{x_k,s} L(x_k, s, \gamma_k, \alpha_k, \beta) = 0$ results in the conditions below:

$$
\begin{array}{ll}
\pi_k - \sum_{i=1}^{m} \lambda_i a_{ik} + \gamma_k + \alpha_k = 0 & \\
\nabla u(s) - \lambda + \beta = 0 & \\
\lambda_i (a_{ik}x_k + s_i - b_i + q_i^{k-1}) = 0 & i = 1, 2, \cdots, m \\
\gamma_k(x_k - 1) = 0 & \\
\alpha_k x_k = 0 & \\
\beta_j s_j = 0 & i = 1, 2, \cdots, m.
\end{array} \tag{9}
$$

Eliminating α_k, β from the above conditions results in the following set of KKT coinditions:

$$
\begin{array}{ll}
\pi_k - \sum_{i=1}^{m} \lambda_i a_{ik} + \gamma_k \le 0 & \\
x_k(\pi_k - \sum_{i=1}^{m} \lambda_i a_{ik} + \gamma_k) & \\
\nabla u(s) - \lambda \le 0 & \\
s_i(\frac{\partial u(s)}{\partial s_i} - \lambda_i) = 0 & i = 1, 2, \cdots, m \\
\gamma_k(x_k - 1) = 0 & \\
\lambda \ free, \gamma_k \le 0, x_k \ge 0, s \ge 0 &
\end{array} \tag{10}
$$

The KKT conditions of the above problem are a subset of KKT conditions of CPCAM. Moreover, the params λ, s are common across the KKT conditions for all values of k. This suggests that the (λ, s) params resulting from solving the problem for k can be used as initial values to solve the problem for $k + 1$.

Another effecive method to solve the SLPM problem, which is inspired by [1], is as follows. Basically in this approach, an LP is solved only for the first bid (iteration). Then, the solution for every bid is used as a basis to find the solution for the next bid using an effecient method that does not require solving the LP.

For $k \ge 2$, suppose the solution of the problem is known for k-1 bids; in particular, the optimal dual price $p(k-1)$ is known. Now the k-th bid arrives which consists of a pair (π_k, a_k). The algorithm handles this bid in three steps as follows:

1. If $\pi_k \le p(k-1)^T.a_k$, then $x_k = 0, p(k) = p(k-1)$ and the algorithm is done for the k-th bid. Otherwise, move to step 2.

2. Set $x_k = 1$. Examine if this can be a solution to the problem as follows. From the feasibility constraints:

$$\begin{aligned} &\sum_{j=1}^{k} a_{ij}x_j + s_i = b_i \quad \forall i \\ \Rightarrow &s_i = b_i - q_i^{k-1} - a_{ij} \quad \forall i \end{aligned} \tag{11}$$

   If for any i, $s_i < 0$, then $x_k = 0, p(k) = p(k-1)$ and the algorithm is done. Further, from the assumption $\frac{\partial u(\cdot)}{\partial s_i}|_{s_i=0} = \infty$, if for any i, $s_i = 0$, then $x_k = 0, p(k) = p(k-1)$ and the algorithm is done. Otherwise, set $p(k) = \nabla u(s)$ with s calculated from (11). In this case, if $\pi_k > p(k)^T.a_k$, then $x_k = 1, p(k) = \nabla u(s)$ are the optimal solution and the algorithm is done. Otherwise, move to step 3.

3. As a result of previous steps, $0 < x_k < 1$. Hence, in the KKT conditions, $\gamma_k = 0$. As a result, $\pi_k = p(k)^T.a_k$. Define $T = \{i|(a_k)_i = 0\}$. Since, for all i, $(a_k)_i \in 0, 1$:

$$\sum_{i \notin T} \frac{\partial u(s)}{\partial s_i} = \pi_k$$

   From the feasibility constraints, define s^* such that $s_i^* = b_i - q_i^{k-1} - a_{ik}x_k$ for all i. Therefore,

$$\sum_{i \notin T} \frac{\partial u(s)}{\partial s_i}|_{s=s^*} = \pi_k$$

   which is a nonlinear equation only in x_k. By solving this equation, $(x_k, p(k))$ can be found. In particular, bisection can be used to solve this non-linear equation. Moreover, based on the nature of $u(\cdot)$, the nonlinear equation might be solvable more efficiently.

3 Question 3

Figures 1,2,3,4 present prices traces for different configuration of the SCPM problem. The same grand truth price $\bar{p}$, as defined below, was used for all the configurations.

$$\bar{p}^T = [1.7483 \quad 1.2746 \quad 1.2939 \quad 1.0666 \quad 1.4282 \quad 1.6447 \quad 1.5057 \quad 1.2919 \quad 1.9765 \quad 1.1826]$$

The figures indicate that the dual prices diverge monotonicaly and also fluctuate. The divergent behavior occurs because of the strictly concave function used in the objective. In particular, with each accepted bid, the dual prices become higher. Moreover, as prices get higher, it is more likely that bids will get rejected. As such, the curve will be flat for longer periods of iteration. The fluctuating behavior occurs because the dual price of each iteration requires to be consistent only with the constraints of the bid of that iteration but not the previous ones.

Furthermore, figures 1,2 indicate by increasing w, fluctuation in the large scale increases while fluctuation in the small scale does not change for $u_1(\cdot)$.

On the other hand, figures 3,4 indicate by increasing w, fluctuation in the large scale increases while fluctuation in the small scale decreases for $u_2(\cdot)$. Furthermore, Figure 4(a) indicates after resources are depleted, the dual prices heavily fluctuate which can be related to the fact that the dual prices are no more required to be unique.

Also the figures indicate that using u_2 will result in the dual prices to grow much larger compared to u_1.

4 Question 4

Figure 5 depicts revenue behavior for different values of k. Figure 5(b) indicates as k increases, the long term revenue increases. Moreover, as k increases, the amount of improvement in revenue decreases.

5 Question 5

The Dynamic Learning Algorithm

Figure 6 depicts the revenue for the Dynamic Learning Algorithm. Comparing to the SPLM method, the dynamic learning algorithm has similar long term overall revenue.

Comparison of SCPM and SLPM

Figure 7 provides the overall revenue behavior for SCPM and SLPM with different parameter values. As the figure indicates, at the beginning SCPM outputperforms SLPM but in the long run, SLPM with with all parameter values surpasses SCPM.

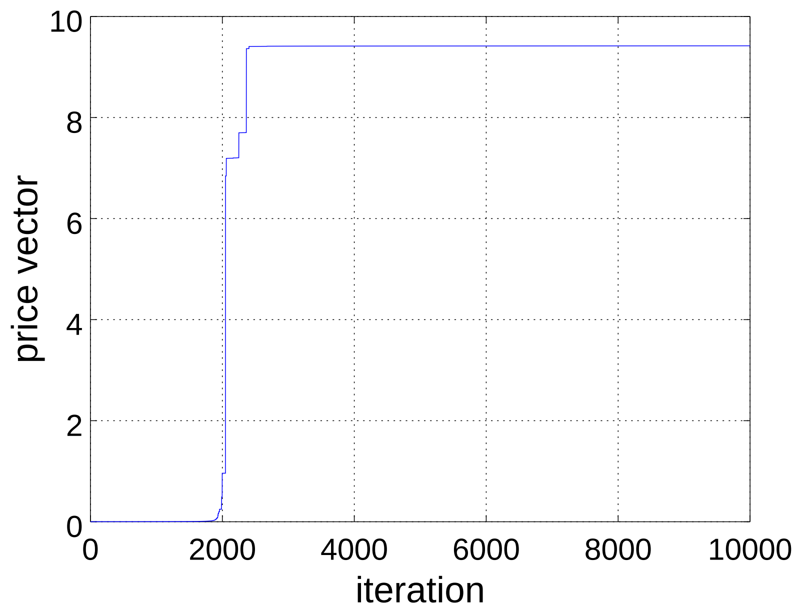
Combination of SCPM and SLPM

The SLPM algorithm requires to solve an LP problem once after k bids (for some constant k) arrive and use the resulting dual prices for decision on future bids. I propose to add a monotone and strictly concave function to the objective, as done in SCPM, and solve this non-linear optimization problem instead of the LP in the SLPM algorithm. This basically combines SLPM and SCPM.

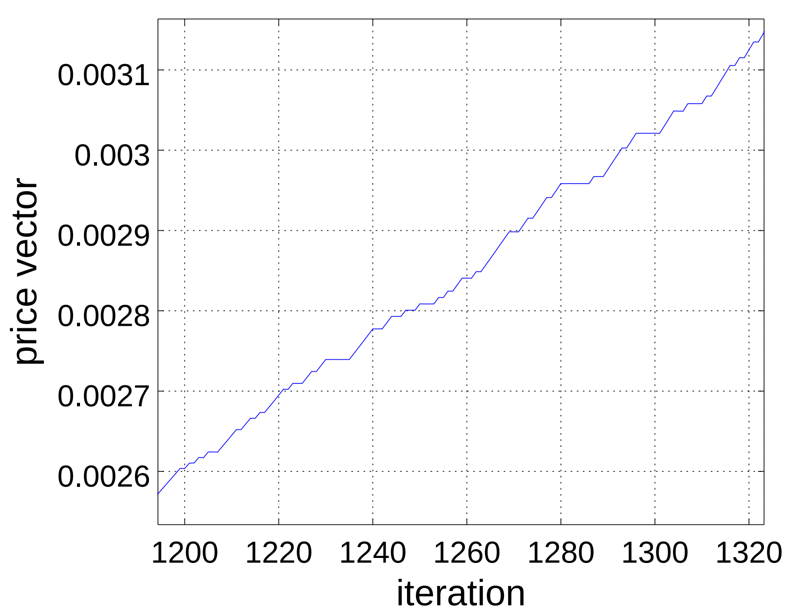
Figure 8 depicts the resulting revenue behavior for the combined algorithm and and different values of k. Similar to Figure 5, Figure 8 indicates as k increases, the long term revenue increases. Moreover, as k increases, the amount of improvement in revenue decreases. On the other hand, the figures indicate the long term revenue produced from the combined algorithm is higher than SLPM.

6 Question 6

In this section, first the version of the problem where a monotone and strictly concave function $u(\cdot)$ is added to the objective is going to be discussed. Then

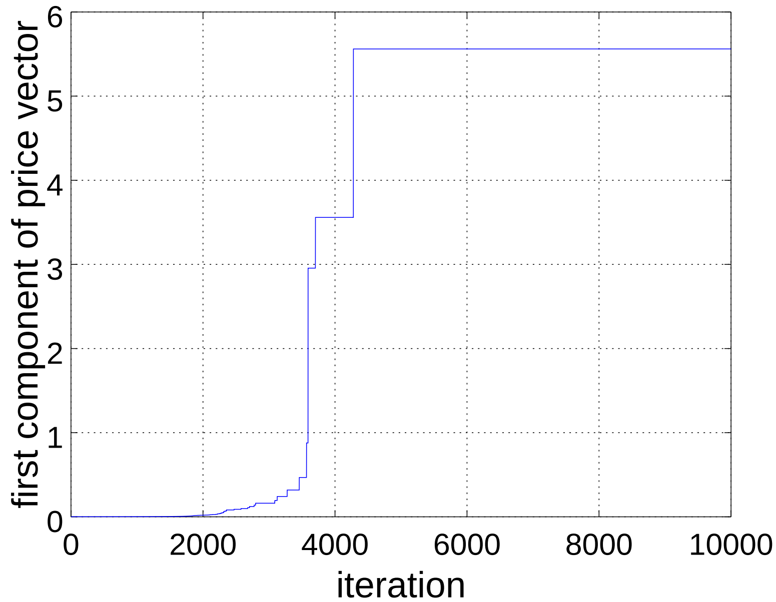


(a) Iteration range: 1-10000 (large scale behavior)

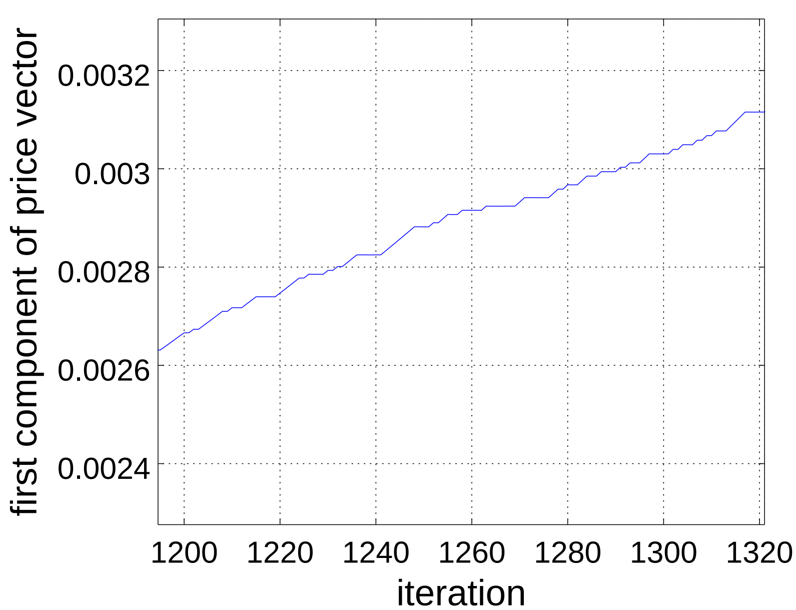


(b) Iteration range: 1200-1300 (small scale behavior)

Figure 1: Dual price behavior for u_1 and $w = 1$

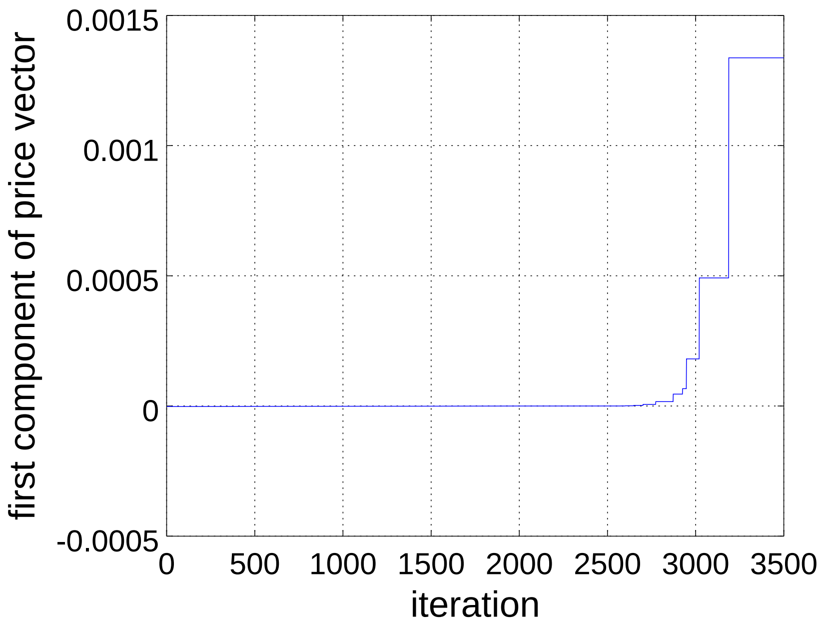


(a) Iteration range: 1-10000 (large scale behavior)

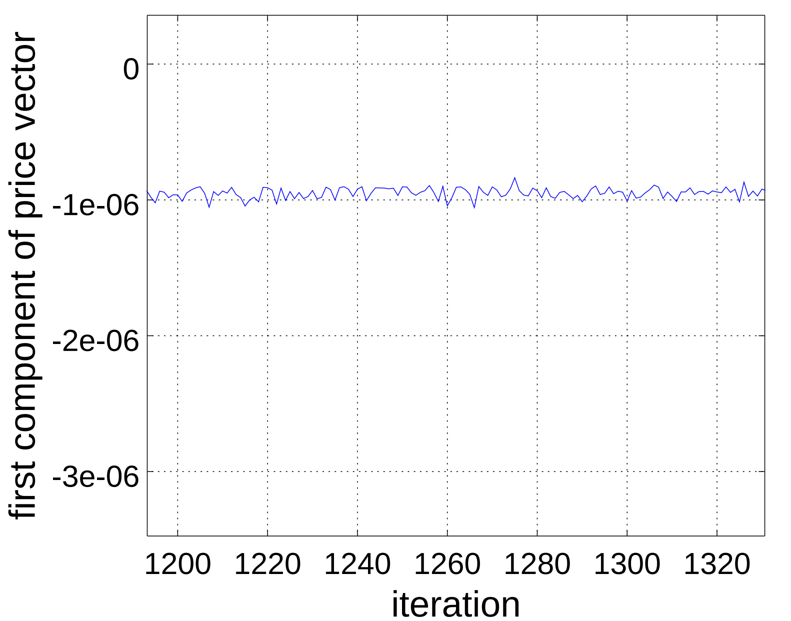


(b) Iteration range: 1200-1300 (small scale behavior)

Figure 2: Dual price behavior for u_1 and $w = 10$

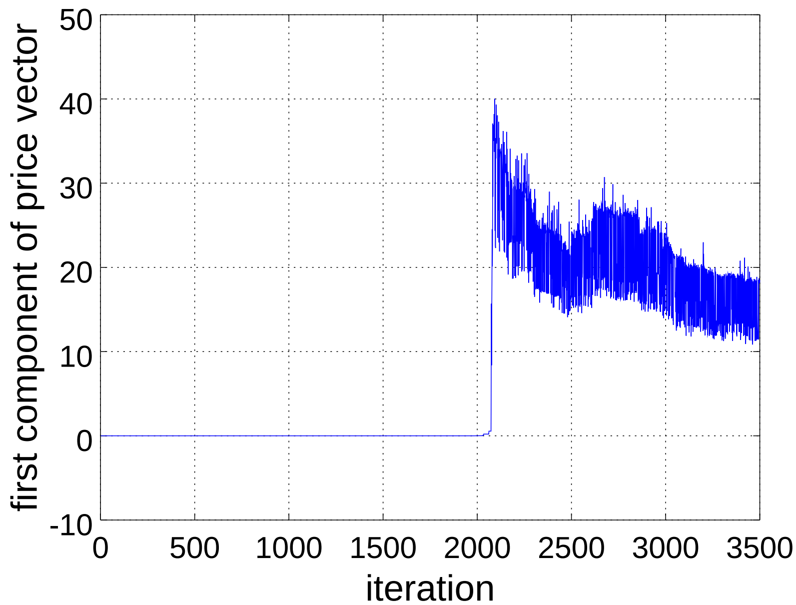


(a) Iteration range: 1-10000 (large scale behavior)

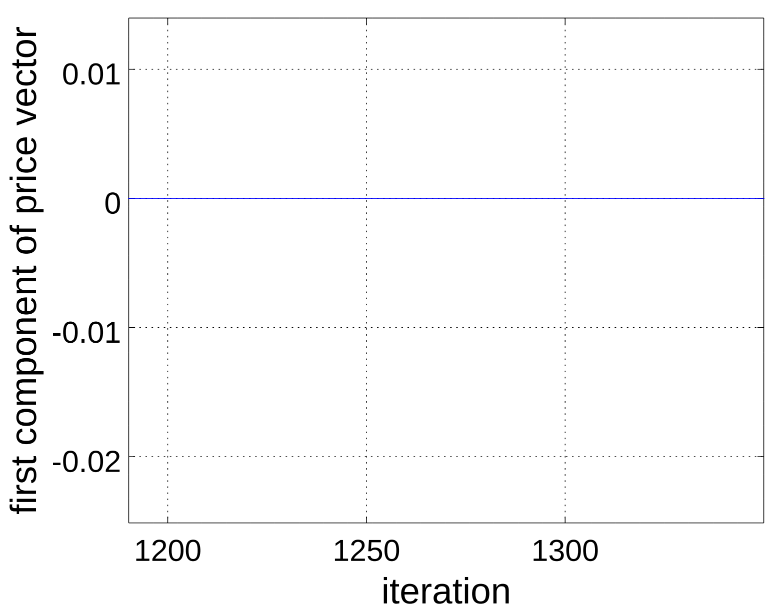


(b) Iteration range: 1200-1300 (small scale behavior)

Figure 3: Dual price behavior for u_2 and $w = 1$

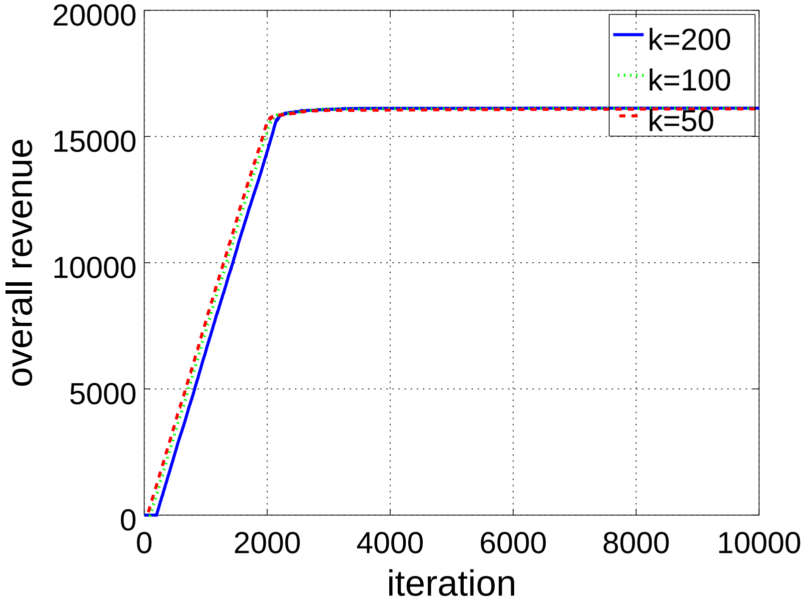


(a) Iteration range: 1-10000 (large scale behavior)

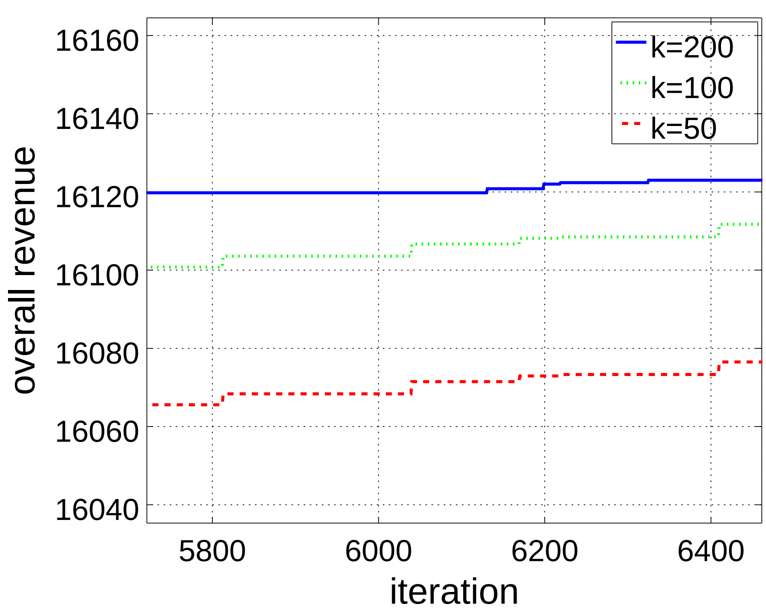


(b) Iteration range: 1200-1300 (small scale behavior)

Figure 4: Dual price behavior for u_2 and $w = 10$



(a) Iteration range: 1-10000 (large scale behavior)



(b) Iteration range: 5800-6400 (small scale behavior)

Figure 5: Revenue vs time for the one time learning algorithm

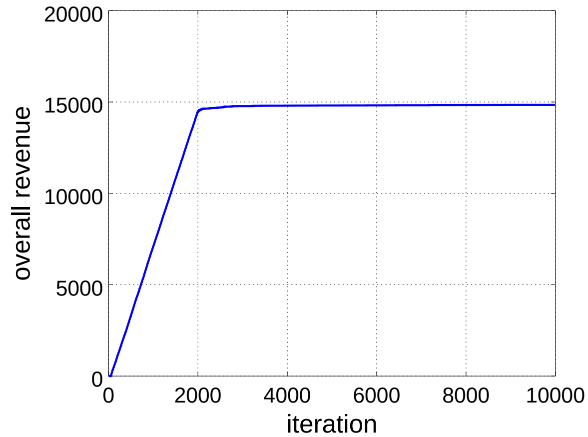


Figure 6: Overall revenue for dynamic learning algorithm

the LP version of the problem (without the strictly convace function) is going to be discussed. At the end, two online learing algorithms will be proposed and studied. Further, to avoid notation confusion, in this section for the production capacities we use vector b.

6.1 Problem including strictly concave objective

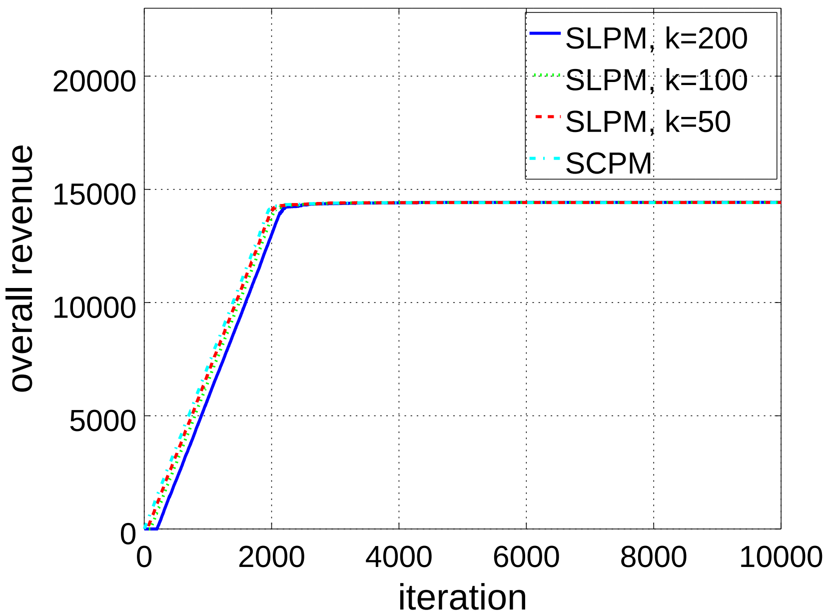
The problem can be rewritten as follows:

$$
\begin{array}{llll}
\min & -(\sum_{j=1}^{n} \pi_j x_j) + \sum_{i,j,k} c_{ijk} y_{ijk} - u(s) & & \\
\text{s.t.} & \sum_k y_{ijk} = a_{ij} x_j & \forall i,j & \lambda_1 \ free \\
 & \sum_{i,j} y_{ijk} + s_k = b_k & \forall k & \lambda_2 \ free \\
 & x_j - 1 \le 0 & j = 1,2,\cdots,n & \gamma \le 0 \\
 & x_j \ge 0 & j = 1,2,\cdots,n & \alpha \ge 0 \\
 & s_k \ge 0 & \forall k & \beta_1 \ge 0 \\
 & y_{ijk} \ge 0 & \forall i,j,k & \beta_2 \ge 0
\end{array} \tag{12}
$$

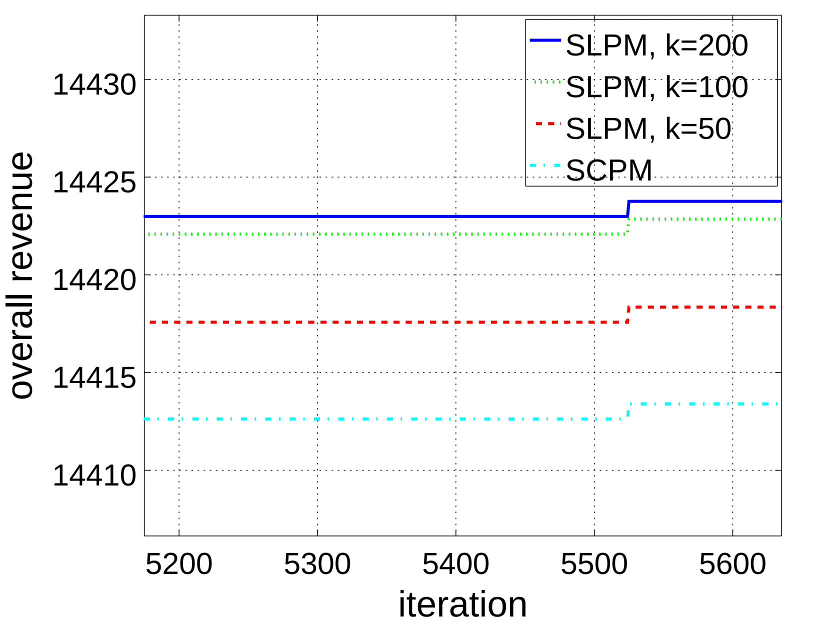
As such, the Lagrangian function is:

$$
\begin{aligned}
L(x,y,s,\lambda_1,\lambda_2,\gamma,\alpha,\beta_1,\beta_2) = & -(\sum_{j=1}^{n} \pi_j x_j) + \sum_{i,j,k} c_{ijk} y_{ijk} - u(s) + \sum_{i,j} (\lambda_1)_{ij} (-\sum_k y_{ijk} + a_{ij} x_j) \\
& + \sum_k (\lambda_2)_k (\sum_{i,j} y_{ijk} + s_k - c_k) \\
& - \sum_{j=1}^{n} \gamma_j (x_j - 1) - \sum_{j=1}^{n} \alpha_j x_j \\
& - \sum_k (\beta_1)_k s_k - \sum_{i,j,k} (\beta_2)_{ijk} y_{ijk}
\end{aligned} \tag{13}
$$

Setting $\nabla_{x,y,s} L(x,s,\lambda_1,\lambda_2,\gamma,\alpha,\beta_1,\beta_2) = 0$ results in the conditions below:

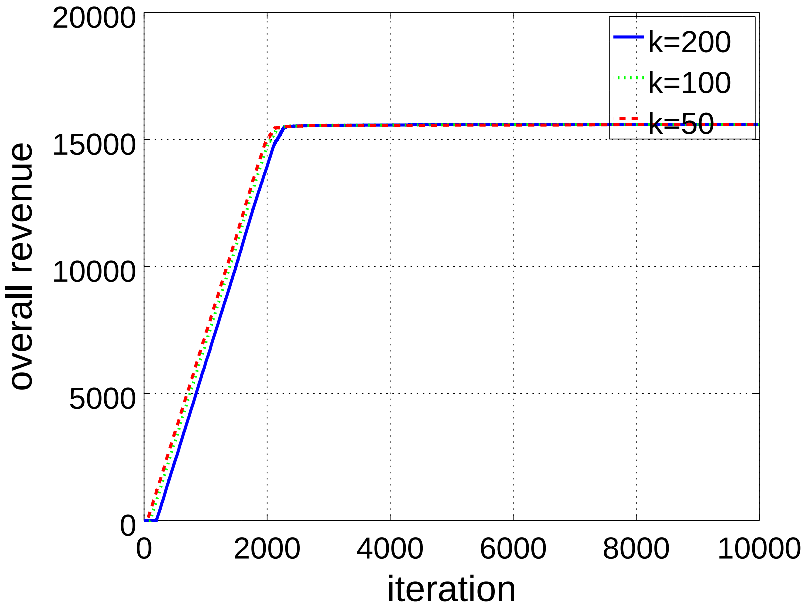


(a) Iteration range: 1-10000 (large scale behavior)

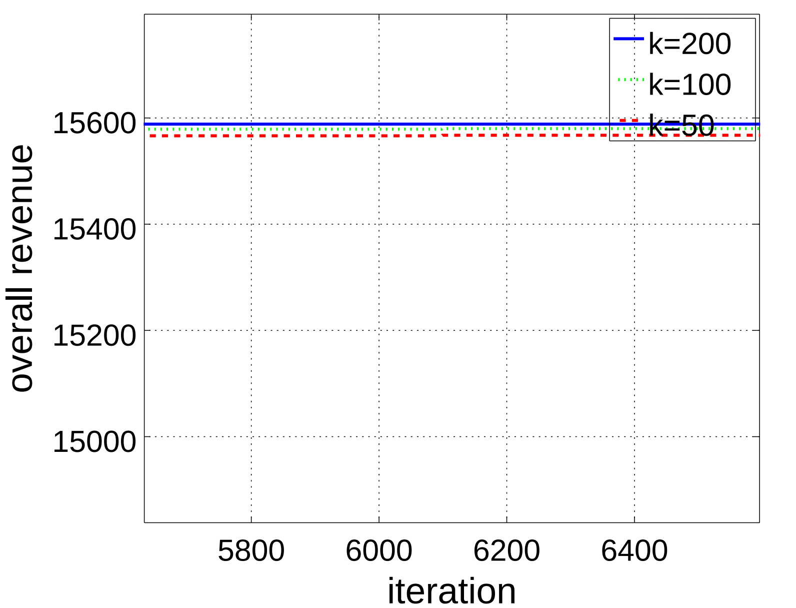


(b) Iteration range: 5200-5600 (small scale behavior)

Figure 7: Comparison of SLPM and SCPM



(a) Iteration range: 1-10000 (large scale behavior)



(b) Iteration range: 5800-6400 (small scale behavior)

Figure 8: Revenue vs time for Combination of SLPM and SCPM

$$
\begin{array}{ll}
-\pi_j + \sum_i (\lambda_1)_{ij} a_{ij} - \gamma_j - \alpha_j = 0 & \forall j \\
c_{ijk} - (\lambda_1)_{ij} + (\lambda_2)_k - (\beta_2)_{ijk} = 0 & \forall i,j,k \\
-\nabla u(s) + \lambda_2 - \beta_1 = 0 & \\
(\lambda_1)_{ij} (\sum_k y_{ijk} - a_{ij} x_j) = 0 & \forall i,j \\
(\lambda_2)_k (\sum_{i,j} y_{ijk} + s_k - b_k) = 0 & \forall k \\
\gamma_j (x_j - 1) & j = 1,2,\cdots,n \\
\alpha_j x_j & j = 1,2,\cdots,n \\
(\beta_1)_k s_k = 0 & \forall k \\
(\beta_2)_{ijk} y_{ijk} = 0 & \forall i,j,k
\end{array}
\tag{14}
$$

Eliminating α, β_1, β_2 from the above conditions results in the following set of KKT coinditions:

$$
\begin{array}{ll}
\pi_j - \sum_i (\lambda_1)_{ij} a_{ij} + \gamma_j \leq 0 & \forall j \\
x_j (\pi_j - \sum_i (\lambda_1)_{ij} a_{ij} + \gamma_j) & \forall j \\
c_{ijk} - (\lambda_1)_{ij} + (\lambda_2)_k \geq 0 & \forall i,j,k \\
y_{ijk} (c_{ijk} - (\lambda_1)_{ij} + (\lambda_2)_k) = 0 & \forall i,j,k \\
\nabla u(s) - \lambda_2 \leq 0 & \\
s_k (\frac{\partial u(s)}{\partial s_k} - (\lambda_2)_k) = 0 & \forall k \\
\gamma_j (x_j - 1) & j = 1,2,\cdots,n \\
\lambda_1 \ free, \lambda_2 \ free, \gamma \leq 0, x \geq 0, s \geq 0 &
\end{array}
\tag{15}
$$

6.1.1 Uniqueness and Non-Negativity

Analysis similar to Question 1 implies for the optimal solution (x^*, y^*, s^*), s^* is unique. Further, assumption $\frac{\partial u(\cdot)}{\partial s_i}|_{s_i=0} = \infty$ implies: $s_i > 0$ for all i. Therefore, from the KKT conditions: $\lambda_2 = \nabla u(s)$. As a result, λ_2 is unique. In addition, since $u(\cdot)$ is increasing, $\lambda_2 \geq 0$.

6.1.2 Interpretation as Transportation Problem

In problem (24), the variables are x, y, s. The structure of the problem suggests it can be divided into two layers. In particular, given x, s, then y is the solution to the LP problem below:

$$
\begin{array}{llll}
v(x,s) := \min & \sum_{i,j,k} c_{ijk} y_{ijk} & & \\
\text{s.t.} & \sum_k y_{ijk} = a_{ij} x_j & \forall i,j & \lambda_1 \ free \\
& \sum_{i,j} y_{ijk} + s_k = c_k & \forall k & \lambda_2 \ free \\
& y_{ijk} \geq 0 & \forall i,j,k &
\end{array}
\tag{16}
$$

As depicted in Figure 9, the problem above is the Transportation problem where every pair (i,j) presents a consumer and c_k for all k presents a supplier. Further, in Figure 9, $d_{i,j} = a_{ij} x_j$, and $d^* = \sum_k c_k - \sum_{i,j} d_{ij}$ presents an extra consumer used to balance demand and supply.

The optimality condition for the above problem is:

$$
(\lambda_1)_{ij} - (\lambda_2)_k \leq c_{ijk}, \quad \forall i,j,k \tag{17}
$$

Furthermore, these are the interpretations of the dual variables:

- $(\lambda_1)_{ij}$ = unit price for consumer (i,j)

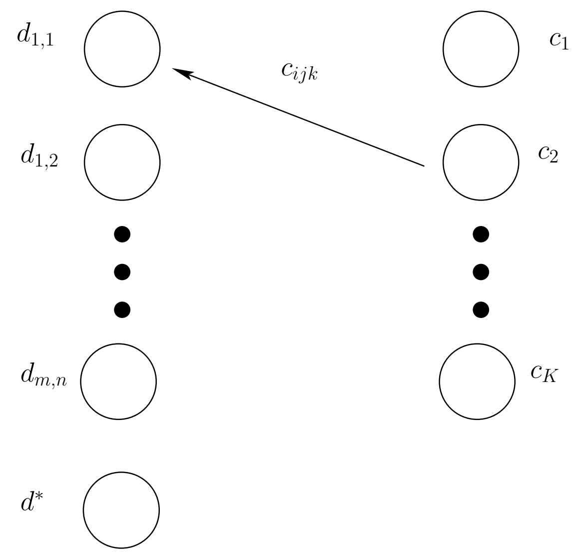


Figure 9: Transportation problem interpretation

- $(-\lambda_2)_k$ = unit price for supplier k

The function $v(x,s)$ defined for (16), is a convex function of $(x,-s)$. Therefore, $v(x,s)$ is a convex function of (x,s). In addition sensitivity analysis of (16) implies:

$$\frac{\partial v}{\partial x_j} = \sum_i a_{ij}(\lambda_1)_{ij}, \quad \forall j \tag{18}$$

$$\frac{\partial v}{\partial s_k} = (-\lambda_2)_k, \quad \forall k \tag{19}$$

Moreover, problem (24) can be restated as follows:

$$\begin{array}{ll} \max & \pi^T x - v(x,s) + u(s) \\ \text{s.t.} & x, s \geq 0 \end{array} \tag{20}$$

So basically, the problem can be solved in two layers. In the first layer (outer layer), the demand (x) and s are decided. Then in the second layer (inner layer), decision is made on which suppliers to use to satisfy demand. It should be noted that in the above problem, the first and second layer are not completeley disconnected. In particualr, the strictly concave function imposes $\lambda_2 \geq 0$ for the subproblem (second layer) concerning $v(x,s)$.

6.2 Problem without concave objective

The problem is:

$$
\begin{array}{llll}
\max & (\sum_{j=1}^{n} \pi_j x_j) - \sum_{i,j,k} c_{ijk} y_{ijk} & & \\
\text{s.t.} & \sum_k y_{ijk} = a_{ij} x_j & \forall i,j & \lambda_1 \ free \\
& \sum_{i,j} y_{ijk} + s_k = b_k & \forall k & \lambda_2 \ free \\
& x_j \leq 1 & j = 1,2,\cdots,n & \gamma \geq 0 \\
& x_j \geq 0 & j = 1,2,\cdots,n & \\
& s_k \geq 0 & \forall k & \\
& y_{ijk} \geq 0 & \forall i,j,k &
\end{array}
\tag{21}
$$

Then, the dual problem is:

$$
\begin{array}{ll}
\min \quad b^T \lambda_2 + e^T \gamma & \\
\pi_j - \sum_i (\lambda_1)_{ij} a_{ij} - \gamma_j \leq 0 & \forall j \\
c_{ijk} - (\lambda_1)_{ij} + (\lambda_2)_k \geq 0 & \forall i,j,k \\
\lambda_2 \geq 0 & \\
\lambda_1 \ free, \gamma \geq 0 &
\end{array}
\tag{22}
$$

where e is the column vector of all ones.

Similar to section 6.1.2, this problem has a Transportation problem interpretation and can be rewritten as follows:

$$
\begin{array}{ll}
\max & \pi^T x - v(x,s) + u(s) \\
\text{s.t.} & x, s \geq 0
\end{array}
$$

Also there is a similar connection between the outer and inner layers.

6.3 Online Algorithm 1

The proposed method below is inspired by the SLPM algorithm. Basically the algorithm below waits for l bids and then solves an LP problem to obtain λ_2. Then, for future bids this is used to make decisions. In this algorithm, there is an assumption to have a good estimate of the total number of bids n.

The algorithm is as follows. For fixed l:

- Set $x_t = 0$ for all $0 \leq t \leq l$.
- Solve the problem below to obtain optimal dual solution $\hat{\lambda}_2$.

$$
\begin{array}{lll}
\max & (\sum_{j=1}^{l} \pi_j x_j) - \sum_{j=1}^{l} \sum_{i,k} c_{ijk} y_{ijk} & \\
\text{s.t.} & \sum_k y_{ijk} = a_{ij} x_j & \forall i, j = 1,2,\cdots,l \\
& \sum_{j=1}^{l} \sum_i y_{ijk} + s_k = \frac{l}{n} b_k & \forall k \\
& x_j \leq 1 & j = 1,2,\cdots,l \\
& x_j \geq 0 & j = 1,2,\cdots,l \\
& s_k \geq 0 & \forall k \\
& y_{ijk} \geq 0 & \forall i, k, j = 1,2,\cdots,l
\end{array}
$$

- For future allocations x_t, solve the problem below to obtain $(\lambda_1)_{*t}$:

$$
\begin{array}{ll}
\min \quad e^T \gamma & \\
\pi_t - \sum_i (\lambda_1)_{it} a_{it} - \gamma_t \leq 0 & \\
(\lambda_1)_{it} \leq c_{itk} + (\hat{\lambda}_2)_k & \forall i, k \\
\lambda_1 \ free, \gamma_t \geq 0 &
\end{array}
\tag{23}
$$

Then,

$$x_t = \begin{cases} 1, & \pi_t - (\lambda_1)^T_{*t} a_t > 0, \text{and satisfying capacity constraints} \\ 0, & \text{otherwise} \end{cases}$$

It should be noted based on the concept of basic feasible solutions, in the optimal solution, for every pair (i, t), there is only one k such that $y_{itk} > 0$. In paritcular, the strictly complementarity condition for the dual problem uniquely identifies for which k, $y_{itk} > 0$. Therefore, checking the capacity constraints will be straigtforward.

Conjecture 1. *For any $\epsilon = \frac{l}{n} > 0$, the algorithm is within $(1 - \epsilon)$ times the optimal value when*

$$B \geq \Omega(f(m, n, K, \epsilon^{-1}))$$

where $B = \min_k b_k$ and $f(\cdot)$ is a monomial (with positive powers) of its arguments.

6.4 Online Algorithm 2

The proposed method below is inspired by the SCPM algorithm. Basically the algorithm below learns a penalty function offline from allocated bids.

At every iteration t of the algorithm, solve the problem below:

$$\begin{array}{llll} \max & \pi_t x_t - \sum_{i,k} c_{itk} y_{itk} + u(s) & & \\ \text{s.t.} & \sum_k y_{itk} = a_{it} x_j & \forall i & \lambda_1 \; free \\ & \sum_i y_{itk} + s_k = b_k - \sum_{j=1}^{t-1} \sum_i y_{ijk} & \forall k & \lambda_2 \; free \\ & x_t \leq 1 & & \\ & x_t \geq 0 & & \\ & s_k \geq 0 & \forall k & \\ & y_{itk} \geq 0 & \forall i, k & \end{array} \tag{24}$$

where $\sum_{j=1}^{t-1} \sum_i y_{ijk}$ presents allocated resources at previous iterations.

As discussed earlier in the section, λ_2 is unique and the problem can be solved in a layered fashion.

The performance of this algorithm with simulated data is provided in Figure 10. For every iteration t of this algorithm, (π_t, c_{itk}, a_t) are randonly generated. In particular, a_t is a random vector in $\{0, 1\}^m$, $\pi_t = p^T a_t + randn(0, 0.2)$, $c_{itk} = randn(0.1, 0.0016)$ and $(b_1, b_2) = (1000, 2000)$.

References

[1] Mark Peters, Anthony So, and Yinyu Ye. Pari-mutuel markets: Mechanisms and performance. *International Workshop on Internet and Network Economics*, pages 82–95, 2007.

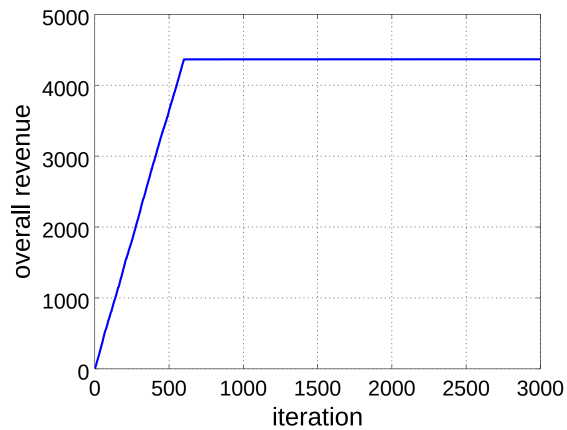


Figure 10: Overall revenue for Learning Algorithm 2