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A Unified Framework for Dynamic Prediction Market Design

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Recently, coinciding with and perhaps driving the increased popularity of prediction markets, several novel pari-mutuel mechanisms have been developed such as the logarithmic market-scoring rule (LMSR), the cost-function formulation of market makers, utility-based markets, and the sequential convex pari-mutuel mechanism (SCPM). In this work, we present a convex optimization framework that unifies these seemingly unrelated models for centrally organizing contingent claims markets. The existing mechanisms can be expressed in our unified framework by varying the choice of a concave value function. We show that this framework is equivalent to a convex risk minimization model for the market maker. This facilitates a better understanding of the risk attitudes adopted by various mechanisms. The unified framework also leads to easy implementation because we can now find the cost function of a market maker in polynomial time by solving a simple convex optimization problem.

In addition to unifying and explaining the existing mechanisms, we use the generalized framework to derive necessary and sufficient conditions for many desirable properties of a prediction market mechanism such as proper scoring, truthful bidding (in a myopic sense), efficient computation, controllable risk measure, and guarantees on the worst-case loss. As a result, we develop the *first* proper, truthful, risk-controlled, loss-bounded (independent of the number of states) mechanism; none of the previously proposed mechanisms possessed all these properties simultaneously. Thus, our work provides an effective tool for designing new prediction market mechanisms. We also discuss possible applications of our framework to dynamic resource pricing and allocation in general trading markets.

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1. Introduction

Contingent claim markets are organized for a variety of purposes. Prediction markets are created to aggregate information about a particular event. Financial markets involving contingent claims allow traders to hedge their exposure to certain event outcomes. Betting markets are designed for entertainment purposes. The participants in these markets trade claims that will pay a fixed amount if a certain event occurs. Some examples of these events would be the winner of the World Series, the value of the latest consumer price index, or the release date of Windows Vista. Prediction markets have grown in popularity as research into the accuracy of their predictions has shown that they effectively aggregate information from the trading population. One of the longest-running prediction markets is the Iowa Electronic Market, which allows real money betting on various elections. Studies by Berg and her coauthors (Berg and Rietz 2006; Berg et al. 2008a, b) have shown that the information generated by these markets often serves as a better prediction of actual outcome than polling data. Google has run internal prediction markets over a variety of events, and Cowgill et al. (2009) have shown that their predictions also perform quite well.

Despite the potential value created by these markets, there can be some difficulties with their introduction and development. First, many nascent markets suffer from liquidity problems. Occasionally these problems stem from the choice of mechanism used to operate the market. Organizing markets as a continuous double auction (like the NASDAQ stock market) is a popular option and usually performs well. However, in thin markets, Bossaerts et al. (2002) have demonstrated that some problems surface that inhibit the growth of liquidity. To overcome this situation, the market organizer could introduce an automated market maker that centrally interacts with the traders. This mechanism follows some prespecified rules for pricing shares. The market organizer must determine these rules with one key question being his own tolerance for risk. Recently, there has been a surge in research of these automated market makers.

New market-making mechanisms based on pari-mutuel principles have recently been developed by Hanson (2003), Pennock (2004), and Peters (2009). These market makers allow contingent claims in nascent markets to be immediately priced according to rules of the mechanism. The mechanisms are pari-mutuel in the sense that the winners are generally paid out by the stakes of the losers. The claims being traded are commitments to pay out a fixed amount if a particular event occurs in the future. The mechanism developed by Hanson has been shown to perform well in simulated markets (see Peters et al. 2007) and has been adopted by many online prediction markets.

However, the origins of these new mechanisms differ. Peters (2009) developed their mechanism by creating a sequential version of a call auction model that is solved by convex optimization. Their sequential convex pari-mutuel mechanism (SCPM) uses a convex optimization problem to determine when to accept orders and how to price accepted orders. On the other hand, Hanson's mechanism is derived from scoring rules. Scoring rules are functions used to compare distributions. In particular, Hanson uses the logarithmic scoring rule to determine how much to charge a trader for a new order. His mechanism is called the logarithmic market-scoring rule (LMSR). Using a similar approach to Hanson's, it is possible to create market makers for other scoring rules. Those market makers are called the market-scoring rule (MSR) market makers. In contrast to the SCPM, the MSR model doesn't directly provide an optimization problem from the market organizer's standpoint.

Recently, there has been some interest in comparing and unifying these mechanisms for prediction markets. Chen and Pennock (2007) give an equivalent cost-function formulation for the MSR market makers, and relate them to utility-based market makers. Peters et al. (2007) empirically compare the performances of various market mechanisms. In this work, we provide a strong theoretical foundation for unifying existing market makers like the SCPM, the MSR, cost-function based markets, and utility-based markets under a single convex optimization framework. Our model not only aids in comparing various mechanisms, but also provides intuitive understanding of the behavior of the market organizer in these seemingly different mechanisms. Specifically, our main contributions are as follows:

- *A unifying framework*: We propose a generalized version of the SCPM as a unified convex optimization framework for market makers. The framework subsumes existing models of prediction market design. In particular, any market-scoring rule, cost-function based market, or utility-based market (of Chen and Pennock 2007) can be expressed into this framework by varying the choice of a concave value function.
- *Intuitive risk-based interpretation*: We establish the equivalence of prediction market mechanisms to the convex risk minimization model for the market maker. The risk attitude of the market maker explains the design choices of various mechanisms. For popular mechanisms like the LMSR, the implicit risk function turns out to characterize precisely how much the market maker is prepared to invest in order to learn a distribution that is different from his prior belief.
- *Value/cost function to scoring rule*: It was shown in Chen and Pennock (2007) that any proper scoring rule implies a cost function defined by certain conditions. However, it had been unknown what type of cost functions would imply proper scoring rules. We establish this direction of the relation by showing that every monotone and concave value function with an onto derivative (easy to check) induces a cost function and implicitly a proper scoring rule. Moreover, the cost function can be constructed by simply solving a single variable convex optimization problem.
- *New mechanism design*: Our framework aids new mechanism design by providing easy-to-check necessary and sufficient conditions for a prediction market mechanism to be *myopically* truthful, proper, loss bounded, and risk measured. As a result, we derive the FIRST loss-bounded, risk-measured, and proper prediction market mechanism that uses a quadratic value function (Quad-SCPM). The original quadratic rule was neither monotone nor risk measured, and the loss of LMSR depends on the number of states. This opens the possibility of handling markets with large number of states.

The rest of the paper is organized as follows. To begin, §2 provides some background on the different mechanisms that we study. In §3, we propose a new framework based on a generalization of the SCPM and demonstrate its truthful pricing property, its corresponding cost function, its guarantee on worst-case loss, and its relation to a convex risk minimization problem for the market maker. In §4, we show that the SCPM framework is equivalent to the cost-function based markets and the MSR market makers, and that it strictly subsumes the set of markets obtainable from the utility framework presented in Chen and Pennock (2007). Section 5 demonstrates how our framework can be used to easily identify key properties of existing mechanisms and develop new mechanisms with desirable characteristics. In §6, we close the paper with a summary of our contributions for prediction market design and implications on design of online trading markets in general.

2. Background

In this section, we provide some background on the key mechanisms for prediction markets discussed in this work—the market scoring rule mechanisms (MSR), the cost-function based market makers, the utility-based market makers, and the sequential convex pari-mutuel mechanism (SCPM).

Let ω represent a discrete or discretized random event to be predicted, with N mutually exclusive and exhaustive outcomes. We consider a contingent claim market where claims are of the form "Pays \$1 if the outcome state is i." A new trader arrives and submits an order that essentially specifies the claims over each outcome state that the trader desires to buy. The market maker then decides what price to charge for the new order. Various mechanisms treat a new order in seemingly different manners.

2.1. Market Scoring Rules

Let $\vec{r} = (r_1, r_2, \ldots, r_N)$ represent a probability estimate for the random event ω. A scoring rule is a sequence of scoring functions, $\mathcal{S} = \mathcal{S}_1(\vec{r}), \mathcal{S}_2(\vec{r}), \ldots, \mathcal{S}_N(\vec{r})$, such that a score $\mathcal{S}_i(\vec{r})$ is assigned to $\vec{r}$ if outcome i of the random variable ω is realized. A proper scoring rule (Winkler 1969) is a scoring rule that motivates truthful reporting of beliefs. Based on proper scoring rules, Hanson (2003) developed the market-scoring rule (MSR) mechanism. In the MSR market, the market maker with a proper scoring rule $\mathcal{S}$ begins by setting an initial probability estimate, $\vec{r}_0$. Every trader can change the current probability estimate to a new estimate of his choice as long as he agrees to pay the market maker the scoring rule payment associated with the current probability estimate and receive the scoring rule payment associated with the new estimate.

Some examples of market scoring rules are the logarithmic market scoring rule (LMSR) (Hanson 2003)

$$\mathcal{S}_i(\vec{r}) = b\log(r_i) \quad (b > 0),$$

and the quadratic market scoring rule

$$\mathcal{S}_i(\vec{r}) = 2br_i - b\sum_j r_j^2 \quad (b > 0).$$

Hanson's MSR has many favorable characteristics. It is designed as a pari-mutuel mechanism that bounds the risk of the market organizer. It functions as an automated market maker in the sense that it is always able to calculate prices for new orders. The LMSR is also known to elicit *truthful* bids from the market traders.

2.2. Cost-Function Based Market Makers

Recently, Chen and Pennock (2007) proposed a cost-function based implementation of market makers. Let the vector $\vec{q} \in \Re^N$ represent the number of claims on each state currently held by the traders. In the cost-function based formulation, the total cost of all the orders $\vec{q}$ is calculated via some cost function $C(\vec{q})$. A trader submits an order characterized by the vector $\vec{a} \in \Re^N$ where a_i reflects the number of claims over state i that he desires. The market organizer charges the new trader $C(\vec{q}+\vec{a}) - C(\vec{q})$ for his order. At any time in the market, the going price of a claim for state i, $p_i(\vec{q})$, equals $\partial C/\partial q_i$. The price is the cost per share for purchasing an infinitesimal quantity of claim i. Chen and Pennock (2007) showed that any scoring rule has an equivalent cost-function formulation. For example, below are the specific cost and pricing functions for LMSR:

$$C(\vec{q}) = b\log\left(\sum_j e^{q_j/b}\right) \quad \text{and} \quad p_i(\vec{q}) = \frac{e^{q_i/b}}{\sum_j e^{q_j/b}}.$$

For general market scoring rules, they proposed three equations that the cost function $C(\cdot)$ should satisfy so that the cost-function based market maker is equivalent to the market based on a given scoring rule $\mathcal{S}$:

$$\begin{aligned} &\mathcal{S}_i(\vec{p}) = q_i - C(\vec{q}) + k_i \quad \forall\, i \\ &\sum_i p_i = 1 \\ &p_i = \frac{\partial C}{\partial q_i} \quad \forall\, i, \end{aligned} \tag{1}$$

where $k_i \in \Re$ can be any constant. We use this formulation later to prove equivalence of the MSR and SCPM mechanisms.

2.3. Utility-Based Market Makers

In Chen and Pennock (2007), the authors also proposed a utility-based market maker. This market maker has a utility function over the final payoffs and maintains his expected utility with respect to a certain subjective probability distribution when running the market. Let $\vec{\theta}$ be the market maker's subjective probability estimate for the outcomes, $u(x)$ be a differentiable mapping $\Re \to \Re$ that expresses his state-independent utility for the final payoff x, and $\vec{m} \in \Re^N$ be the vector of payoffs across all states. At each stage of the market, the risk-neutral price of each state is defined by

$$p_i = \frac{\theta_i u'(m_i)}{\sum_j \theta_j u'(m_j)},$$

where $u'(x)$ is the derivative of $u(\cdot)$ at x.

Chen and Pennock (2007) showed that using this price, the utility-based market maker preserves the utility at a certain constant level during the whole trading process, namely,

$$\sum_j \theta_j u(m_j) = k,$$

where $k \in \Re$ is a constant. They also show that the worst-case loss for the market maker is bounded under some regularity conditions on $u(\cdot)$. They defined the cost function for the utility-based market maker as the solution of the following equation,

$$\sum_j \theta_j u(C(\vec{q}) - q_j) = k,$$

and showed that for a special class of utility functions (i.e., the HARA utility functions), the utility-based market maker is equivalent to a market maker with a corresponding pseudospherical scoring rule. Chen and Pennock were the first to formulate the problem in terms of the market maker's risk attitude with respect to future payoffs. Unfortunately, in practice the expressiveness of their model is limited because market makers are typically not highly informed about the probability of each outcome and are therefore unable to commit to a choice of subjective probabilities.

2.4. Sequential Convex Pari-Mutuel Mechanism

The SCPM was designed to require traders to submit orders that include three elements: a limit price $\pi \in \Re$, a limit quantity l, and a vector $\vec{a}$ that describes the order. Specifically, each component of the vector $\vec{a}$ either takes the value of 1 (if a claim over the specified state is desired) or 0 (if it is not desired). The limit price refers to the maximum amount that the trader wishes to pay for one share. The limit quantity represents the maximum number of shares that the trader is willing to buy. The market maker decides the actual number of shares x of a new order to grant and the price to charge for it. The market maker makes this decision by solving the following optimization problem:

$$\begin{aligned} \underset{x,z,\vec{s}}{\text{maximize}} \quad & \pi x - z + \sum_i \beta_i \log(s_i) \\ \text{s.t.} \quad & \vec{a}x + \vec{s} + \vec{q} = z\vec{e} \\ & 0 \leqslant x \leqslant l, \end{aligned} \tag{2}$$

where parameter $\vec{q}$ stands for the numbers of shares held by the traders prior to the arrival of the new order $(\pi, l, \vec{a})$, and $\vec{e}$ represents the vector of all ones. Each time a new order arrives, the optimization problem (2) is solved and the state prices, denoted by $\vec{p}$, are defined to be the optimal-dual variables associated with the first set of constraints. The trader is then charged according to the inner product of the state price vector and the order filled, i.e., $\vec{p}^T\vec{a}$.

This optimization problem has the following interpretation for the market maker. The variable z represents the largest number of accumulated shares for any of the outcomes, whereas for any i, s_i represents the contingent numbers of surplus shares kept by the market maker given that the state is i. The function $v(\vec{s}) = \sum_i \beta_i \log(s_i)$, with $\vec{\beta} \geqslant 0$, captures the "future value" of these surplus shares. When $\vec{\beta} = 0$, the objective reduces to maximizing $\pi x - z$, which represents the worst-case profit made after accepting the new order. Later in this paper, we establish that the market maker can adjust his risk attitude through choosing $\vec{\beta}$.

2.5. Other Market Makers

The dynamic pari-mutuel market (DPM) of Pennock (2004) is another popular market maker that we should mention. It was created as a cost-function based market maker where the cost and price functions are derived from some desired ratios between prices and the number of shares granted in each state. In particular, the following cost and pricing functions are commonly used:

$$C(\vec{q}) = \kappa \sqrt{\sum_j q_j^2} \quad \text{and} \quad p_i(\vec{q}) = \frac{\kappa q_i}{\sqrt{\sum_j q_j^2}}.$$

A key difference between this mechanism and the other mechanisms considered in this paper is that the DPM doesn't guarantee a fixed payoff of \$1 for each winning share. Although the value of a winning order is lower bounded by κ, its exact value is not known until the last order is submitted. Mainly due to this issue, we will not be considering this market maker in the remainder of this work.

3. The Unifying Framework

In this paper, we illustrate that a generalized formulation of the SCPM provides a unifying framework for dynamic prediction market design. We propose the following convex optimization model with a *concave* continuous value function $v(\vec{s})$:[1]

$$\begin{aligned} \underset{x,z,\vec{s}}{\text{maximize}} \quad & \pi x - z + v(\vec{s}) \\ \text{s.t.} \quad & \vec{a}x + \vec{s} + \vec{q} = z\vec{e} \\ & 0 \leqslant x \leqslant l. \end{aligned} \tag{3}$$

Note that the original SCPM model of Peters et al. (2007) is a special case of (3) that uses the value function $v(\vec{s}) = \sum_i \beta_i \log s_i$. From here on, "SCPM" refers to the above generalized SCPM model. When required, we disambiguate by referring to the original model with $v(\vec{s}) = \sum_i \beta_i \log s_i$ as "Log-SCPM." The optimization model (3) has exactly the same meaning for the market maker as the Log-SCPM, and inherits many desirable properties like intuitive interpretation, convex formulation, global optimality, Lagrange duality, polynomial computational complexity, etc., from the original Log-SCPM model. Next, we demonstrate some new desirable properties of the new SCPM framework, including truthfulness of the pricing scheme, efficient cost-function based scoring rules, easily computable guarantees on worst-case loss, and a risk measure interpretation.

3.1. Truthful Pricing Scheme

In this section, we design a truthful pricing scheme for the generalized SCPM framework. We shall assume limited misreports, that is, we assume that a bidder may only lie about his valuation π of an order. There are no misreports of the arrival time of the bidder in the market, and a bidder is only allowed to bid once. Such truthfulness is also known as myopic truthfulness.

The original implementation of the SCPM model does not provide incentives for the traders to bid truthfully (Peters et al. 2007, Peters 2009). On the other hand, market-scoring rules such as LMSR ensure myopically truthful bidding. We show that this difference is attributed to a difference between the implementation of the SCPM and the market-scoring rule-pricing scheme. As explained in §2.4, in the original SCPM model, the market organizer charged a new trader based on the dual state price vector, which was sensitive to the bid value of the trader. However, in the market-scoring rules such as LMSR, the trader is charged according to a cost function, which is independent of the current trader's bid.

In what follows, we show that our general SCPM framework is equivalent to the generalized VCG mechanism (Groves 1973, also see Nisan et al. 2007), and the VCG pricing scheme gives a general truthful pricing scheme for this model. In fact, as we show later, the VCG pricing scheme is equivalent to charging the traders based on the cost function of the market maker. This gives an interesting interpretation of the cost-function based implementation of these markets.

Theorem 1. *Irrespective of the choice of value function $v(\cdot)$, the SCPM mechanism admits myopically truthful bidding under VCG pricing scheme.*

Proof. The optimization problem (3) used by the SCPM model to decide the allocation x^* for a new incoming trader can be rewritten as

$$x^* = \arg\max_{\{0\leqslant x\leqslant l\}} \pi x + \chi(x),$$

where $\chi(x)$ is the concave function defined as

$$\chi(x) = \underset{z,\vec{s}}{\text{maximize}} \quad -z + v(\vec{s}) \qquad \text{s.t.} \quad \vec{a}x + \vec{s} + \vec{q} = z\vec{e}.$$

This allocation method is an "affine maximizer" (affine in the bid value π). A truthful pricing mechanism for such model is given by generalized Vickrey-Clarke-Groves (VCG) scheme where the trader is charged an amount $\chi(0) - \chi(x^*)$, i.e., the externality the agent imposes on other agents (Nisan et al. 2007). □

Additionally, this connection to the VCG scheme allows us to extend truthful trading to richer betting markets. This is illustrated in the following example where multiple orders are processed simultaneously.

Example 1 (Multiple Orders). Instead of one trader at a time, let multiple traders with orders $(\vec{a}_i, \pi_i, l_i)_{i=1}^k$ be allowed to enter the market simultaneously. These orders can actually be processed and charged together. Specifically, the market maker will solve the following program:

$$\begin{aligned} \underset{x_i,z,\vec{s}}{\text{maximize}} \quad & \sum_{i=1}^{k} \pi_i x_i - z + v(\vec{s}) \\ \text{s.t.} \quad & \sum_{i=1}^{k} \vec{a}_i x_i + \vec{s} + \vec{q} = z\vec{e} \\ & 0 \leqslant x_i \leqslant l_i \quad \forall i. \end{aligned}$$

Suppose the optimal solution is $(\vec{x}^*, \vec{s}^*, z^*)$. Then the VCG scheme, which ensures truthful bidding, prices each order i by $\chi_i(0) - \chi_i(x_i^*)$, where function $\chi_i(x_i)$ is defined as

$$\begin{aligned} \chi_i(x_i) := \underset{x_j,z,\vec{s}}{\text{maximize}} \quad & \sum_{j\neq i} \pi_j x_j - z + v(\vec{s}) \\ \text{s.t.} \quad & \sum_{j\neq i} \vec{a}_j x_j + \vec{a}_i x_i + \vec{s} + \vec{q} = z\vec{e} \\ & 0 \leqslant x_j \leqslant l_j, \quad \forall j \neq i. \end{aligned}$$

Remark 1. Note that the concept of "truthfulness" is only concerned with the design of bidding and pricing mechanism so that a rational and myopic trader bids his true valuation for an order that is fixed ("$\vec{a}$" is fixed). Theorem 1 allows us to establish a clear connection between SCPM and the theory of mechanism design for bidding markets (Nisan et al. 2007) where "truthfulness" plays an important role. In §4.2, we will also make the connection with scoring rule markets and the notion of "properness." Intuitively, properness of a scoring rule further ensures that it is profitable for traders to buy securities until the "market belief" reflects their actual belief, thus, it relates directly to the choice of orders "$\vec{a}$."

3.2. Cost Function of the Market Maker

Section 2.2 discussed a cost-function based implementation for the market makers, introduced by Chen and Pennock (2007). The cost function $C(\vec{q})$ for the market maker represents the total money collected by the market organizer and depends only on the total orders $\vec{q}$ allocated so far. An incoming trader with order $\vec{a}$ is charged a price of $C(\vec{q}+\vec{a}x) - C(\vec{q})$.

An easily computable convex cost function for the SCPM market with VCG pricing can be derived as follows.

Lemma 1. *Let $\vec{q}$ be the number of shares on each state held by the traders in an SCPM market with value function $v(\cdot)$. The cost function $C(\vec{q})$ associated with this market is given by*

$$C(\vec{q}) = \min_t t - v(t\vec{e} - \vec{q}). \tag{4}$$

The cost function $C(\vec{q})$ is convex, arbitrage free, and has the property that $\vec{e}^T\nabla C(\vec{q}) = 1,\ \forall \vec{q} \geqslant 0$.

Proof. To prove that (4) is the cost function for this market maker, simply note that $C(\vec{q}+\vec{a}x) - C(\vec{q}) = \chi(0) - \chi(x)$, i.e., the VCG price for the incoming order.

$C(\vec{q})$ is convex because it is the minimum over t of a function that is jointly convex in t and $\vec{q}$. It is easy to verify the no-arbitrage condition $C(\vec{q}+t\vec{e}) = t + C(\vec{q})$ from the definition of $C(\vec{q})$ in (4). Finally, the property that $\vec{e}^T\nabla C(\vec{q}) = 1,\ \forall \vec{q} \geqslant 0$ follows from the optimality condition for this optimization problem. □

It is to be noted that not only is the cost function convex and arbitrage free, but it can also be easily computed for any given $\vec{q}$ by solving a single-variable convex optimization problem. This is in contrast to the market-scoring rule markets, where computing the cost function is nontrivial and requires solving the set of differential equations shown in Equation (1).

3.3. Worst-Case Loss for the Market Maker

An interesting consequence of the cost function representation of the SCPM is that the worst-case loss can be formulated as a convex optimization problem:

Theorem 2. *Assuming the market starts with zero shares initially, then the worst-case loss for the market maker using the SCPM mechanism is given by* $B + C(0)$ *where*

$$B = \max_i \left\{ \max_{\vec{s}} v(\vec{s}) - s_i \right\}$$

and $C(\cdot)$ *is the cost function defined by* (4).

Proof. Let the number of shares held by the traders at a certain moment be $\vec{q}$. By Lemma 1, assuming we started with zero shares, the total money collected by that time is $C(\vec{q}) - C(0)$. On the other hand, if state i occurs, the market maker needs to pay amount q_i. Thus, the worst-case loss of the market maker when state i occurs can be found by solving the optimization problem

$$\begin{aligned}&\max_{\vec{q}\geqslant 0} q_i - (C(\vec{q}) - C(0))\\ &\quad= \max_{t,\,\vec{q}\geqslant 0} \{(q_i - t) + v(t\vec{e} - \vec{q})\} + C(0)\\ &\quad= \max_{\vec{s}} \{v(\vec{s}) - s_i\} + C(0).\end{aligned}$$

We finally take the maximum among all outcome states to conclude the proof. □

The following corollary states the condition under which the market maker has bounded loss:

Corollary 1. *Computing the worst-case loss for the SCPM is a convex optimization problem. Furthermore, a necessary and sufficient condition to guarantee a bounded loss is that for all i and* $\vec{s}$, $v(\vec{s}) - s_i$ *is bounded from above.*

Below, we illustrate the applications of the above theorem through some examples. Detailed proofs for these examples are available in Appendix A.

Example 2. Let the min-SCPM market be the one with $v(\vec{s}) = \min_i s_i$. For this market, the worst-case loss is 0 because $C(0) = \min_t \{t - v(t\vec{e})\} = 0$, and for all i, $v(\vec{s}) - s_i \leqslant 0$. Observe that in this case, the market maker is maximizing the worst-case profit, which represents extreme risk averseness.

Example 3. Let the exponential-SCPM market with uniform prior be the one with $v(\vec{s}) = b\sum_i (1/N) \cdot (1 - e^{-s_i/b})$. We can derive the cost function to be $C(\vec{q}) = b\log(\sum_i (1/N)\exp(q_i/b))$ and verify that $B = b\log N$, $C(0) = 0$, giving a worst-case loss of $b\log N$.

Example 4. For the log-SCPM, $v(\vec{s}) = \sum_i \beta_i \log(s_i)$, if $\beta_i > 0$ for some i, then we can show that B is unbounded by setting $s_1 = 1$, $s_i = \alpha$, and letting $\alpha \to \infty$. We have:

$$B \geqslant \lim_{\alpha\to\infty} \beta_1 \log 1 + \sum_{i\neq 1} \beta_i \log(\alpha) - 1 = \infty$$

and $C(0) = \sum_i \beta_i - \sum_i \beta_i \log \sum_i \beta_i$. Thus, the worst-case loss is unbounded in this case.

These examples provide a glimpse of how the value function encodes the risk averseness of the market maker. In §3.4, we further confirm these insights by recasting our optimization framework as a risk minimization problem.

3.4. Risk Minimization Formulation for SCPM

Each time he or she is offered an order, the market maker must consider the risks involved in accepting it. This is due to the fact that the monetary return generated from the market depends on the actual outcome. In the earlier pari-mutuel market introduced in Lange and Economides (2005), this risk was effectively handled in terms of maximizing the worst-case return generated by the market relative to the set of outcomes (i.e., $v(\vec{s}) = \min_i s_i$, refer to Example 2). Unfortunately, this risk attitude is somewhat limiting because it leads to a market that is likely to accept very few orders and extract little information. In what follows, we consider the return generated by the market to be a random variable Z and demonstrate that when he uses SCPM with a nondecreasing value function, the market maker effectively takes rational decisions with respect to a risk attitude. We use duality theory to gain new insights about how this attitude relates to the concept of prior belief of market maker about the probability of outcomes.

In a finite, discrete probability space $(\Omega, \mathcal{F})$, the set of random variables $\mathcal{Z}$ can be described as the set of functions $Z\colon \Omega \to \Re$. A convex risk measure on the set $\mathcal{Z}$ is defined as follows:

Definition 1. When the random variable Z represents a return, a risk measure is a function $\rho\colon \mathcal{Z} \to \Re$ that describes one's attitude towards risk as: random return Z is preferred to Z' if $\rho(Z) \leqslant \rho(Z')$. Furthermore, a risk measure is called convex if it satisfies the following:

- *Convexity*: $\rho(\lambda Z + (1-\lambda)Z') \leqslant \lambda\rho(Z) + (1-\lambda)\rho(Z')$, $\forall Z, Z' \in \mathcal{Z}$, and $\forall \lambda \in [0, 1]$.
- *Monotonicity*: If $Z, Z' \in \mathcal{Z}$ and $Z \geqslant Z'$, then $\rho(Z) \leqslant \rho(Z')$.
- *Translation Equivariance*: If $\alpha \in \Re$ and $Z \in \mathcal{Z}$, then $\rho(Z + \alpha) = \rho(Z) - \alpha$.

Convex risk measures are intuitively appealing. First, even in a context where the decision maker does not know the probability of occurrence for the different outcomes, it is still possible to describe a risk function $\rho(Z)$. The three properties of convex risk measures are also natural ones to expect from such a function. Convexity states that diversifying the returns leads to lower risks. Monotonicity states that if the returns are reduced for all outcomes, then the risk is higher. And finally, translation equivariance states that if a fixed income is added to the random return, then it is irrelevant whether this fixed income is received before or after the random return is realized. We refer the reader to Föllmer and Schied (2002) for a deeper study of convex risk measures.

Next, we formulate the SCPM model for prediction markets as a convex risk minimization problem. In context of prediction markets, let Z represent the random return for the market organizer, which depends on the actual outcome of the random event in question. Let $\vec{q}$ represent the total orders held by the traders, and c represent the total money collected so far from the traders in the market. Because the market organizer has to pay \$1 for each accepted order that matches the outcome, his return for outcome state i is $c - q_i$. When a new trader enters with a bid of π, based on the number of accepted orders x, the total return for state i is given by $(c - q_i + \pi x - a_i x)$. The risk minimization model seeks to choose the number of accepted orders x to minimize the risk on total return. Below, we formally show that the SCPM model is equivalent to a convex risk minimization model.

Theorem 3. *Let $\Omega = \{\omega_1, \omega_2, \ldots, \omega_m\}$, $\vec{Z} \in \Re^m$ be the vector representation of Z such that $\vec{Z}_i = Z(\omega_i)$, and $Z^x(\omega_i) = c - q_i + \pi x - a_i x$. Then, given that $v(\cdot)$ is nondecreasing, the SCPM optimization model* (3) *is equivalent in terms of the set of the optimal solutions for x to the risk minimization model*

$$\begin{aligned}\underset{x}{\text{minimize}}\quad & \rho(Z^x)\\ \text{s.t.}\quad & 0 \leqslant x \leqslant l\end{aligned}$$

with convex risk measure $\rho(Z) = \min_t \{t - v(\vec{Z} + t\vec{e})\}$.

Proof. The equivalence can be obtained by first eliminating $\vec{s}$ in (3), and then performing a simple change of variable $t = z - \pi x - c$:

$$\begin{aligned}\max_z\ & \pi x - z + v\big(z\vec{e} - \vec{a}x - \vec{q}\big)\\ &= \max_t -t + v\big(t\vec{e} + (\pi x + c)\vec{e} - \vec{a}x - \vec{q}\big) - c\\ &= -\min_t \big\{t - v(\vec{Z}^x + t\vec{e})\big\} - c\\ &= -\rho(Z^x) - c.\end{aligned}$$

Because maximizing $-\rho(Z^x) - c$ over x is equivalent to minimizing $\rho(Z^x)$ in terms of optimal solution set, the equivalence follows directly.

It remains to show that when $v(\cdot)$ is concave and nondecreasing, the proposed measure satisfies the three properties (convexity, monotonicity, and translational equivariance) of a convex risk measure. The convexity and the monotonicity follow directly from concavity and monotonicity of $v(\cdot)$. We refer the reader to Appendix B for more details on this part of the proof. □

Remark 2. More importantly, Theorem 3 can be used to show that *any* convex risk measure $\rho(Z)$ can potentially be used to create a version of the SCPM market that accepts orders according to the risk attitude described by $\rho(Z)$. This is achieved by simply choosing the value function $v(\vec{s}) = -\rho(Y^{\vec{s}})$ where $Y^{\vec{s}}: \Omega \to \Re$ is a random variable defined as $Y^{\vec{s}}(\omega_i) = s_i$. Such a constructed $v(\cdot)$ is necessarily concave and increasing.

We just showed that the SCPM market actually represents a risk minimization problem for the market maker when $v(\cdot)$ is nondecreasing. In fact, we can get more insights about the specific risk attitude by studying the dual representation of risk measure $\rho(Z)$:

$$-\rho(Z) = \min_{\vec{p} \in \{\vec{p} \mid \vec{p} \geqslant 0,\, \sum_i p_i = 1\}} \mathbb{E}_{\vec{p}}[Z] + \mathscr{L}(\vec{p}),$$

where $\mathscr{L}(\vec{p}) = \max_{\vec{s}} v(\vec{s}) - \vec{p}^T\vec{s}$, and $\mathbb{E}_{\vec{p}}[Z] = \sum_i p_i \vec{Z}_i$. We refer the reader to Föllmer and Schied (2002) for more details on the equivalence of this representation. Note that $\rho(Z)$ is evaluated by considering the worst distribution $\vec{p}$ in terms of trading off between reducing expected return and reducing the penalty $\mathscr{L}(\vec{p})$.

In terms of the SCPM, this representation equivalence leads to the conclusion that orders are accepted according to

$$\max_{0 \leqslant x \leqslant l} \left(\min_{\vec{p} \in \{\vec{p} \mid \vec{p} \geqslant 0,\, \sum_i p_i = 1\}} \sum_i p_i \vec{Z}_i^x + \mathscr{L}(\vec{p}) \right).$$

In this form, it becomes clearer how $\mathscr{L}(\vec{p})$ encodes the intents of the market maker and relates it to his belief about the true distribution of outcomes. For instance, we know that the first order is accepted only if

$$\forall \vec{p} \in \left\{\vec{p} \mid \vec{p} \geqslant 0, \sum_i p_i = 1\right\}, \quad \sum_i p_i \vec{Z}_i^x \geqslant -(\mathscr{L}(\vec{p}) - \mathscr{L}(\hat{p})),$$

where $\hat{p} = \arg\min_{\vec{p} \in \{\vec{p} \mid \vec{p} \geqslant 0,\, \sum_i p_i = 1\}} \mathscr{L}(\vec{p})$. For any given $\vec{p}$, the penalty $\mathscr{L}(\vec{p}) - \mathscr{L}(\hat{p})$ therefore reflects how much the market maker is willing to lose in terms of expected returns in the case that the true distribution of outcomes ends up being $\vec{p}$. It is also the case that after accepting $\vec{q}$ orders, the distribution described by $\vec{p}^* = \arg\min_{\vec{p}} \vec{p}^T(c\vec{e} - \vec{q}) + \mathscr{L}(\vec{p})$ is actually the vector of dual prices computed in the SCPM. In other words, the state price vector in the SCPM market reflects the distribution that is being considered as the outcome distribution by the market organizer in order to determine his expected return. This confirms the interpretation of prices as a belief consensus on outcome distribution generated from the market.

As we will see next, the function $\mathcal{L}(\vec{p})$ is typically chosen so that $\mathcal{L}(\vec{p}) - \mathcal{L}(\hat{p})$ is large if $\vec{p}$ is far from $\hat{p}$, and $\hat{p}$ reflects a prior belief of the market organizer. That is, the market organizer is willing to lose some of his expected return in order to learn a distribution $\vec{p}$ that is very different from his prior belief. This is in accordance with the fact that the market is being organized as a prediction market rather than a pure financial market, and one of the goals of a market organizer is to learn beliefs even if at some risk of generating less returns.

We make the above interpretation clearer through the following examples.

Example 5. In the min-SCPM market, the value function $v(\vec{s}) = \min_i\{s_i\}$ corresponds to cost $\mathcal{L}(\vec{p}) = \max_s\{\min_i\{s_i\} - \vec{p}^T\vec{s}\} = 0$ for all $\vec{p}$. That is, the market organizer is purely maximizing his worst-case return.

Example 6. In the Exponential-SCPM market with uniform prior, the value function $v(\vec{s}) = b\sum_i(1/N)(1 - \exp(-s_i/b))$ corresponds to the penalty function $\mathcal{L}(\vec{p}) = b\mathcal{L}_{\mathrm{KL}}(\vec{p}\|U)$, which is the Kullback-Leibler divergence of $\vec{p}$ from uniform distribution. This is minimized at $\vec{p} = U$ reflecting a uniform prior. The corresponding risk measure is also known as the entropic risk measure and its level of tolerance to risk is measured by b. Later, we show that this model is actually equivalent to the popular LMSR model.

Example 7. The log-SCPM uses $v(\vec{s}) = \sum_i \beta_i \log(s_i)$, which is equivalent to choosing the penalty function to be $\mathcal{L}(\vec{p}) = b\mathcal{L}_{\mathrm{LL}}(\vec{p}\|\vec{\theta})$ where $b = \sum_i \beta_i$, $\theta_i = \beta_i / \sum_i \beta_i$, and $\mathcal{L}_{\mathrm{LL}}(\vec{p}\|\vec{\theta}_i) = -\log(\prod_i p_i^{\theta_i}) + k$. More specifically, $\mathcal{L}(\vec{p})$ can be interpreted as the negative log-likelihood that $\vec{p}$ is the true distribution given a set of observations described by the vector $\vec{\beta}$. This penalty function is minimized at $\vec{p} = \vec{\theta}$ and tolerance to risk is measured by b.

These examples illustrate how the risk minimization representation provides insights on how to choose $v(\cdot)$. In the case of Example 5, the associated penalty function leads to a market where trades that might generate a loss for the market maker are necessarily rejected. Hence, the traders have no incentive for sharing their belief. On the other hand, both the value function for the exponential-SCPM and log-SCPM lead to mechanisms that accept orders leading to negative expected returns under a distribution $\vec{p}$, as long as this distribution is "far enough" from $\hat{p}$. Effectively, a trader with a belief that differs from $\hat{p}$ will have his order accepted given that he submits it early enough. In practice, choosing between the exponential-SCPM and the log-SCPM involves determining whether the Kullback-Leibler divergence or a likelihood measure better characterizes the market maker's commitment to learning the true distribution.

Remark 3. In some of these examples, the value function actually takes the "expected value" form, i.e., $v(\vec{s}) = \sum_i \theta_i u(s_i)$ for some probability vector $\vec{\theta} > 0$ such that $\vec{e}^T\vec{\theta} = 1$ and some one-dimensional increasing concave mapping $u(\cdot)$. This is the case for the log-SCPM market, the exponential-SCPM market, and the linear-SCPM market discussed later. When discussing the risk interpretation of these cases, one needs to mention that they are all members of a family of risk measures called optimized certainty equivalent and introduced by Ben-Tal and Teboulle (2007). In practice, this family of convex risk measures have become popular in many applied fields partly because they can be easily defined and because the associated risk minimization problem can be approximated using Monte Carlo methods when the size of the outcome space becomes large (or even infinite).

4. Relationship to Existing Mechanisms

In the last section, we showed that the SCPM, with properly chosen value function, possesses many desirable properties for designing a prediction market. The properties include myopic truthfulness for the traders, bounded loss for the market makers, and a controllable risk measure interpretation. In the following, we establish the relationship between the SCPM and existing mechanisms. We show that many existing mechanisms, including the market-scoring rules and the utility-based market makers, are subsumed by our SCPM framework.

4.1. Relationship to the Cost-Function Based Markets

The cost-function based markets in their original formulation do not involve bids (π) or limit (l) as in the SCPM. A trader simply observes the current price and demands a quantity w of his order $\vec{a}$. The market organizer grants these orders at price $C(\vec{q}+w\vec{a}) - C(\vec{q})$. We show that these markets are equivalent to the SCPM market in the following sense:

Theorem 4. *Any arbitrage-free cost-function based market with convex cost function $C(\vec{q})$ can be simulated as an SCPM market with value function $v(\vec{s}) = -C(-\vec{s})$. Conversely, any SCPM market with value function $v(\cdot)$ can be simulated as an arbitrage-free cost-function based market with convex cost function $C(\vec{q}) = \min_t t - v(t\vec{e} - \vec{q})$.*

Proof. Consider a cost-function based market with convex cost function $C(\cdot)$. Because the market is arbitrage free, $C(\vec{q} + t\vec{e}) = t + C(\vec{q})$ for any t. Let $\vec{q}$ denotes the currently held shares. Consider an incoming trader who wishes to buy w quantity of order $\vec{a}$. In the equivalent SCPM market, set $v(\vec{s}) = -C(-\vec{s})$. And, the new trader's order is $(\pi, l, \vec{a})$ with $\pi = \nabla C(\vec{q}+\vec{a}w)^T\vec{a}$ and $l = w$. To see that this SCPM market simulates the original cost-function based market, firstly note that because $C(\cdot)$ is convex, $v(\cdot)$ is a concave function. Also, due to the no-arbitrage condition, $-z + v(\vec{s}) = -z - C(-\vec{s}) = -C(z\vec{e} - \vec{s})$. Therefore, the SCPM market decides the quantity of orders x to grant using the following optimization problem (refer to (3)):

$$\begin{aligned} \underset{x,z,\vec{s}}{\text{maximize}} \quad & \pi x - C(z\vec{e} - \vec{s}) \\ \text{s.t.} \quad & z\vec{e} - \vec{a}x - \vec{s} = \vec{q} \\ & 0 \leqslant x \leqslant w. \end{aligned}$$

This is a convex optimization problem with KKT conditions:

$$\vec{p}^T\vec{a} + y \geqslant \pi, \quad x\cdot(\vec{p}^T\vec{a} + y - \pi) = 0,$$
$$p_i = \nabla C(\vec{q} + \vec{a}x)_i, \quad y\cdot(w - x) = 0, \quad y \geqslant 0,\ 0 \leqslant x \leqslant w.$$

Thus, x is increased until $x = w$, or the price $\vec{p}^T\vec{a} = \nabla C(\vec{q} + \vec{a}x)^T\vec{a}$ becomes greater than the bid price π. Because $\pi = \nabla C(\vec{q} + \vec{a}w)^T\vec{a}$, the optimal solution for x is $x^* = w$, i.e., the number of orders accepted is the same as in the market based on the cost function $C(\vec{q})$. Also, the instantaneous price is $\vec{p} = \nabla C(\vec{q} + \vec{a}x)$, which is the same as in the cost-function based market.

To see that the price charged is the same, recall that SCPM charges using the VCG scheme, where an incoming trader with allocation x is charged $C_{\text{SCPM}}(\vec{q} + \vec{a}x) - C_{\text{SCPM}}(\vec{q})$ with $C_{\text{SCPM}}(\vec{q})$ defined as (refer to Lemma 1)

$$C_{\text{SCPM}}(\vec{q}) = \min_t t - v(t\vec{e} - \vec{q}).$$

For the proposed value function $v(\vec{s}) = -C(-\vec{s})$, this reduces to

$$C_{\text{SCPM}}(\vec{q}) = \min_t t - C(-t\vec{e} + \vec{q}) = \min_t t - t + C(\vec{q}) = C(\vec{q}).$$

Thus, at any point the two markets accept the same quantity of orders and charge the same price to an incoming trader.

Similarly, to prove the other direction, consider an SCPM market with value function $v(\cdot)$, and an incoming trader with order $(\pi, l, \vec{a})$. In the equivalent cost-function based market, $C(\vec{q}) = \min_t t - v(t\vec{e} - \vec{q})$ and the trader would demand x^* quantity of order $\vec{q}$, where x^* is the optimal solution for the SCPM market. Following the same line of argument as above, one can see that the two markets are equivalent. □

4.2. Relationship to the Market Scoring Rule

The market-scoring rules (MSR) form a large class of popular pari-mutuel mechanisms. In this section, we demonstrate a strong equivalence between the SCPM and the market-scoring rule markets. Particularly, we show that:

Theorem 5. *Any proper market-scoring rule with cost function $C(\cdot)$ can be formulated as an SCPM market maker* (3) *with the concave value function $v(\vec{s}) = -C(-\vec{s})$, and the two market makers are equivalent in terms of the orders accepted and the price charged for submitted orders.*

Theorem 6. *The SCPM with any value function $v(\cdot)$ gives an implicit proper scoring rule, as long as the value function has the property that its derivative spans the simplex $\{\vec{r}\colon \vec{e}^T\vec{r} = 1,\ \vec{r} \geqslant 0\}$, that is, for all vectors $\vec{r}$ in the simplex*

$$\exists \vec{s}, \quad \nabla v(\vec{s}) = \vec{r}. \tag{5}$$

Further, the implicit scoring rule is strictly proper if the function $v(\cdot)$ is smooth over the simplex.

Thus, the SCPM framework subsumes the class of proper scoring rule mechanisms. Moreover, a proper scoring rule based market can be created by simply choosing a value function $v(\cdot)$ that satisfies condition (5). As we shall demonstrate later in this section, this condition is not difficult to satisfy or validate, thus provides a useful tool to design market mechanisms that correspond to a proper scoring rule.

We first prove the above theorems for general market scoring rule based market, and then illustrate with two specific examples: the LMSR and the quadratic market scoring rules.

4.2.1. Equivalence Between SCPM and the Market-Scoring Rules (Theorem 5). To establish Theorem 5, we use the cost-function formulation of market-scoring rules discussed in Chen and Pennock (2007), and briefly explained in §2.2. Theorem 4 established equivalence of the SCPM to any cost-function based market as long as the cost function is convex and satisfies the no-arbitrage condition. Below we show that the cost function of any proper scoring rule automatically satisfies these conditions. Thus, Theorem 5 follows directly from Theorem 4 and the lemma below.

Lemma 2. *The cost function $C(\cdot)$ for any proper scoring rule has following properties*:

1. *$C(\vec{q})$ is a convex function of $\vec{q}$.*
2. *For any vector $\vec{q}$ and scalar d, it holds that*: *$C(\vec{q} + d\vec{e}) = d + C(\vec{q})$.*

Proof. To prove that $C(\vec{q})$ is convex, it suffices to show that for any $\vec{q}_0$ and $\vec{q}_1$, $C(\vec{q}_0 + \lambda\vec{q}_1)$ is convex in λ. We have the following

$$\frac{dC(\vec{q}_0 + \lambda\vec{q}_1)}{d\lambda} = \vec{q}_1 \cdot \nabla C|_{(\vec{q}_0 + \lambda\vec{q}_1)} = \vec{q}_1 \cdot p(\vec{q}_0 + \lambda\vec{q}_1).$$

Therefore, it suffices to show that $\vec{q}_1 \cdot \vec{p}(\vec{q}_0 + \lambda\vec{q}_1)$ is increasing in λ. In what follows, we denote $\vec{p}(\vec{q}_0 + \lambda\vec{q}_1)$ by $\vec{p}(\lambda)$ and $C(\vec{q}_0 + \lambda\vec{q}_1)$ by $C(\lambda)$. By the properness of $\mathscr{S}$ and the first condition of (1), we have that for any λ_1 and λ_2,

$$\begin{aligned} &\sum_i p_i(\lambda_1)\mathscr{S}_i(\vec{p}(\lambda_1)) \geqslant \sum_i p_i(\lambda_1)\mathscr{S}_i(\vec{p}(\lambda_2)) \\ &\quad\Rightarrow\ \sum_i p_i(\lambda_1)\big((\vec{q}_0 + \lambda_1\vec{q}_1)_i - C(\lambda_1) + k_i\big) \\ &\qquad \geqslant \sum_i p_i(\lambda_1)\big((\vec{q}_0 + \lambda_2\vec{q}_1)_i - C(\lambda_2) + k_i\big) \\ &\quad\Rightarrow\ (\lambda_1 - \lambda_2)\big(\vec{q}_1 \cdot \vec{p}(\lambda_1)\big) \geqslant C(\lambda_1) - C(\lambda_2). \end{aligned} \tag{6}$$

Similarly,

$$\sum_i p_i(\lambda_2)\mathcal{S}_i(\vec{p}(\lambda_2)) \geqslant \sum_i p_i(\lambda_2)\mathcal{S}_i(\vec{p}(\lambda_1))$$
$$\Rightarrow\ (\lambda_2-\lambda_1)(\vec{q}_1\cdot\vec{p}(\lambda_2)) \geqslant C(\lambda_2)-C(\lambda_1). \tag{7}$$

Then (6) and (7) yield

$$(\lambda_2-\lambda_1)\big(\vec{q}_1\cdot(\vec{p}(\lambda_2)-\vec{p}(\lambda_1))\big)\geqslant 0,$$

which guarantees that $\vec{q}_1\cdot\vec{p}(\lambda_2)\geqslant\vec{q}_1\cdot\vec{p}(\lambda_1)$ when $\lambda_2\geqslant\lambda_1$. Thus, $C(\vec{q})$ is convex in $\vec{q}$ for every cost function corresponding to a proper scoring rule.

We prove part 2 by contradiction. Assume that there exists $\vec{q}$ and d such that $C(\vec{q}+d\vec{e}) > d + C(\vec{q})$. Set $\vec{q}\,' = \vec{q}+d\vec{e}$. Then, $\mathcal{S}_i(\vec{p})-\mathcal{S}_i(\vec{p}\,') = (q_i-q_i')-(C(\vec{q})-C(\vec{q}\,'))>0$ for all i, implying that $\sum_i p_i'(\mathcal{S}_i(\vec{p})-\mathcal{S}_i(\vec{p}\,'))>0$, which contradicts the properness of scoring rule $\mathcal{S}$. Similarly, we can prove a contradiction for $C(\vec{q}+d\vec{e})<d+C(\vec{q})$. Note that in Chen and Pennock (2007), the second property was treated as an assumption based on the principle of no arbitrage. Here, we show that it can actually be derived from the properties of the cost-function formulation itself. □

4.2.2. Properness of the SCPM (Theorem 6). In this section, we prove that the SCPM mechanism is proper, that is, it implicitly corresponds to scoring the reported beliefs with a proper scoring rule, as long as the value function satisfies the spanning condition (5). Intuitively, in order to ensure that the trader can report his true belief vector, which can take any value in the simplex, it is necessary to ensure that price $\nabla v(\vec{s})$ can be set to that value for some $\vec{s}$. Thus, $\nabla v(\vec{s})$ must span the simplex. Below, we rigorously prove that this condition is both necessary and sufficient to ensure properness.

Proof of Theorem 6. By definition, a scoring rule $\mathcal{S}(\cdot)$ is proper if and only if, given that the true belief is $\vec{r}$, it is optimal for a selfish trader to report belief, that is,

$$\vec{r}\in\arg\max_{\vec{p}}\sum_i r_i\mathcal{S}_i(\vec{p}).$$

In a cost-function based market, the traders do not directly report a belief $\vec{p}$. Instead, they buy shares $\vec{q}$ paying a price equal to the difference of the cost function, and thus they indirectly report the belief through the final price vector $\vec{p}$. For these markets, an implicit scoring rule is defined in the following manner (Chen and Pennock 2007):

$$\mathcal{S}_i(\vec{p}) = q_i - C(\vec{q}) + k_i \quad \forall\, i,$$

where q is such that $p_i = \partial C/\partial q_i\ \forall\, i$ (also refer to Equation (1) in §2.2). Therefore, the properness condition in terms of $\vec{q}$ is represented as

$$\vec{p}^* := \nabla C(\vec{q}^*) = \vec{r}, \tag{8}$$

$$\text{where } \vec{q}^* \in \arg\max_{\vec{q}\geqslant 0}\sum_i r_i(q_i - C(\vec{q})) \tag{9}$$

for all distributions $\vec{r}$.

Intuitively, because the traders receive \$1 for each share of the actual outcome, the profit of traders under outcome state i is $q_i - C(\vec{q})$. Thus, the "properness" condition ensures that an optimal strategy for selfish traders is to buy orders $\vec{q}$ so that the resulting price vector $\vec{p}$ is equal to their actual belief $\vec{r}$. Now, the optimality conditions for (9) are

$$r_i - \frac{\partial C(\vec{q}^*)}{\partial q_i^*} + \eta_i^* = 0, \quad \eta_i^*\geqslant 0,\ q_i^*\geqslant 0,$$
$$\eta_i^* q_i^* = 0, \quad \forall\, i.$$

Thus, condition (8) is satisfied if there exists a positive optimal solution to (9). As derived in Lemma 1, the cost function of the SCPM mechanism is given by (4). Therefore, the optimization problem (9) is equivalent to

$$\begin{aligned}\max_{\vec{q}\geqslant 0}\sum_i r_i(q_i - C(\vec{q})) &\equiv \max_{\vec{q}\geqslant 0}\sum_i r_i\Big(q_i - \min_t\{t - v(t\vec{e}-\vec{q})\}\Big)\\ &\equiv \max_{\vec{q}\geqslant 0,\, t}\vec{r}^{\,T}(\vec{q}-t\vec{e}) + v(t\vec{e}-\vec{q})\\ &\equiv \max_{\vec{s}:\,\vec{s}=t\vec{e}-\vec{q},\,\vec{q}\geqslant 0} v(\vec{s}) - \vec{r}^{\,T}\vec{s}\\ &\equiv \max_{\vec{s}} v(\vec{s}) - \vec{r}^{\,T}\vec{s}.\end{aligned}$$

As long as there exists an optimal solution $\vec{s}^*$ to the above problem, we can set t^* as a large positive value and set $\vec{q}^* = t^*\vec{e}-\vec{s}^* > 0$. Thus, condition (5), which requires that $\nabla v(\vec{s})$ spans the simplex, ensures the properness. One can also show that this is also a necessary condition. This concludes our proof of Theorem 6. □

A concern, however, is that the price vector $\vec{p}^*$ that maximizes the trader's expected profit may not be unique. This could be either because there are multiple subgradients of the cost function $C(\vec{q})$ at optimal $\vec{q}^*$ resulting in multiple price vectors $\vec{p}^*$, or because there are multiple optimal $\vec{q}^*$ and they all result in different corresponding price vectors $\nabla C(\vec{q}^*)$. This is typically undesirable, because in this case, either buying the orders $\vec{q}^*$ associated with the true belief $\vec{r}$ is not the only optimal strategy for the traders, or even in the case that the traders acquire $\vec{q}^*$, the market maker is still unable to recover the true belief. This situation is avoided by the concept of strictly proper scoring rules. A scoring rule is called *"strictly proper"* if the *only* optimal strategy for traders is to honestly report the belief (Winkler 1969). In terms of our market mechanism, it means that the optimal price vector $\vec{p}^*$ that satisfies conditions (8) and (9) must be unique. Because $v(\cdot)$ is concave, it is easy to see that a sufficient condition to ensure strict properness in the SCPM is that $v(\cdot)$ is a smooth function, that is, $\nabla v(\cdot)$ is continuous (over the simplex).

Example 8. For LMSR market maker, the cost function is known to be $C(\vec{q}) = b\log(\sum_i e^{q_i/b})$ (Chen and Pennock 2007). Thus, this market is equivalent to the SCPM framework with value function $v(\vec{s}) = -C(-\vec{s}) = -b\log(\sum_i e^{-s_i/b})$. This scoring rule is known to be strictly proper (Hanson 2003). Note that our condition for properness is satisfied as well since $v(\cdot)$ is smooth and

$$\nabla v(\vec{s}) = \left[\frac{e^{-s_i/b}}{\sum_i e^{-s_i/b}}\right],$$

which clearly spans the simplex.

Example 9. A market maker using quadratic scoring rule is equivalent to the SCPM framework with value function $v(\vec{s}) = -C(-\vec{s}) = \vec{e}^T\vec{s}/N - (1/4b)\vec{s}^T(I - (\vec{e}\vec{e}^T/N))\vec{s}$. This scoring rule is known to be strictly proper (Chen and Pennock 2007). Our condition for properness is satisfied since $v(\cdot)$ is smooth and

$$\nabla v(\vec{s}) = \left[\frac{1}{N} + \frac{\bar{s} - s_i}{2b}\right],$$

where $\bar{s} = \vec{e}^T\vec{s}/N$. Thus, for any $\vec{r}$ in simplex, we can set $s_i = -2br_i$ to get $\nabla v(\vec{s}) = \vec{r}$.

Further examples from existing and new markets appear in §5.

4.3. Relationship to the Expected Utility Framework

In §2.3, we briefly discussed the utility-based market model introduced in Chen and Pennock (2007). In this section, we show that their utility-based market maker model is strictly subsumed by our SCPM model. Our derivation is constructive in the sense that we provide the means of constructing an SCPM model that replicates the behavior intended from a utility-based market.

Theorem 7. *Any expected-utility based market maker represented by the triplet $(\vec{\theta}, u, k)$ such that $\vec{\theta} \geqslant 0$, $\sum_j \theta_j = 1$, $u(x)$ is a nondecreasing concave utility function and $\inf_x u(x) < k < \sup_x u(x)$ can be formulated as an SCPM model with the concave nondecreasing value function $v(\vec{s})$ defined as the optimal value of the convex optimization problem*:

$$v(\vec{s}) := \underset{t}{\text{maximize}} \quad t \tag{10a}$$

$$\text{s.t.} \quad \sum_j \theta_j u(s_j - t) \geqslant k. \tag{10b}$$

Proof. We first show that $v(\vec{s})$ is a concave nondecreasing function of $\vec{s}$ and satisfies the conditions required in the SCPM framework. Problem (10) is necessarily a convex problem because it has a linear objective and a feasible region that is convex because $u(x)$ is concave. By inserting Lagrange multipliers, we can equivalently represent this value function as

$$v(\vec{s}) = \max_t \min_{\lambda > 0} t + \lambda\left(\sum_j \theta_j u(s_j - t) - k\right).$$

In this form, we can see that $v(\vec{s})$ is concave because it is the maximum over t of a function that is concave jointly in $\vec{s}$ and t; this last fact is due to the inner term being the minimum of concave functions that is known to be concave. We also know that $v(\vec{s})$ is nondecreasing, because for each fixed $\lambda > 0$ and t, the function $t + \lambda(\sum_j \theta_j u(s_j - t))$ is nondecreasing in $\vec{s}$.

Based on Lemma 1 and Theorem 4, we know that the SCPM market with $v(\vec{s})$ is equivalent to a cost-function based market that uses the cost function

$$C_{\text{SCPM}}(\vec{q}) = \min_r r - v(re - \vec{q})$$
$$= \inf\left\{r - t \,\middle|\, \sum_j \theta_j u(r - q_j - t) \geqslant k\right\} = -v(-\vec{q}).$$

We also know from Chen and Pennock (2007) that the expected utility framework behaves exactly as a cost-function based market with $C_{\text{UF}}(\vec{q})$ defined as the unique mapping that satisfies: $\sum_j \theta_j u(C_{\text{UF}}(\vec{q}) - q_j) = k$. Thus, if we can show that $\sum_j \theta_j u(C_{\text{SCPM}}(\vec{q}) - q_j) = k$, or more specifically that $\sum_j \theta_j u(-v(-\vec{q}) - q_j) = k$, then it is clear that the expected utility framework market is replicated by a SCPM market that uses $v(\vec{s})$ as defined in (10).

Let x_1 and x_2 be values such that $u(x_1) < k < u(x_2)$; these values exist because k was assumed to be in the interior of the range of $u(\cdot)$. For any $\vec{q}$, because $u(x)$ is nondecreasing, we know that for $t_1 = -x_1 - \min_j q_j$ and $t_2 = -x_2 - \max_j q_j$, it is necessary that $\sum_j \theta_j u(-t_1 - q_j) < k < \sum_j \theta_j u(-t_2 - q_j)$. Because $u(x)$ is continuous by definition, the intermediate value theorem guarantees us that there exists a $\bar{t} \in [t_2, t_1]$ such that $\sum_j \theta_j u(-\bar{t} - q_j) = k$. We conclude that the optimal value of Problem (10) with $\vec{s} = -\vec{q}$, let's call it $t^* = v(-\vec{q})$, is such that $\sum_j \theta_j u(-t^* - q_j) = k$; otherwise, the expected utility with t^* would need to be strictly greater than k and, because $u(x)$ is nondecreasing, $\bar{t}$ would necessarily outperform t^*. □

Example 10. In Chen and Pennock (2007), the authors proposed using the *hyperbolic absolute risk aversion* (HARA) class of utility functions to represent the market maker's risk aversion in the context of a subjective probability vector $\vec{\theta}$. Based on the result presented in Theorem 7, we know that the SCPM mechanism with

$$v(\vec{s}) := \underset{t}{\text{maximize}}\, t$$

$$\text{s.t.} \sum_i \theta_i\left(\frac{1}{1-\gamma}\left(\gamma\Big(M + \frac{\alpha}{\gamma}(s_i - t)\Big)^{1-\gamma} - 1\right)\right) \geqslant k,$$

where M is a real number, γ is an extended real number, and $\alpha > 0$ implements an equivalent market. The theory developed for the SCPM mechanism also tells us that for any member of the HARA, one can compute its associated worst-case loss bound by solving a convex optimization problem.

We just showed that the utility-based market makers are subsumed by the market makers in the SCPM framework. In the following example, we show that the reverse is not true.

Lemma 3. *The three-outcome SCPM market defined by the value function*

$$v(\vec{s}) = \max_{\vec{d}\leqslant\vec{s}} -\vec{d}^T A\vec{d} + \frac{\vec{e}^T\vec{d}}{3} \quad \text{with } A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

cannot be replicated in the utility framework of Chen and Pennock (2007), *described in* §2.3.

Proof. After some transformation, we have that the SCPM market with such a value function behaves according to the cost-function based market with

$$C(\vec{q}) := \underset{t,\vec{d}}{\text{minimize}} \quad t + \vec{d}^T A\vec{d} - \frac{\vec{e}^T\vec{d}}{3} \tag{11a}$$

$$\text{s.t.} \quad \vec{d} \leqslant t\vec{e} - \vec{q}. \tag{11b}$$

Now we compute the prices at $\vec{q} = 0$ and $\vec{q}' = [1,1,0]^T$, respectively. By construction of the cost function, we know that the instantaneous prices generated at $\vec{q}$ must be equal to the optimal assignment of dual variables associated with constraint (11b). Thus, $(\vec{p}, \vec{q})$ should satisfy the KKT conditions:

$$\vec{e}^T\vec{p} = 1$$

$$2A\vec{d} - \frac{\vec{e}}{3} + \vec{p} = 0$$

$$\vec{d} \leqslant t\vec{e} - \vec{q}$$

$$\vec{p} \geqslant 0$$

$$\vec{p}^T(\vec{d} - t\vec{e} + \vec{q}) = 0.$$

We can show that $\{\vec{q} = 0, \vec{d} = 0, t = 0, \vec{p} = \vec{e}/3\}$ and $\{\vec{q} = [1,1,0]^T, \vec{d} = [1,1,7/6]^T, t' = 2, \vec{p} = [1/3, 2/3, 0]^T\}$ satisfy these conditions, and the price is unique for both $\vec{q}$. Therefore, we know that $\vec{p}(0) = \vec{e}/3$ and $\vec{p}([1,1,0]^T) = [1/3, 2/3, 0]^T$. We now show by contradiction that there exists no expected utility framework that can replicate these prices.

Assuming that there exists an expected utility framework that can replicate such a cost-function based market, then it must accept the same orders and generate the same instantaneous prices. In the utility framework, according to Chen and Pennock (2007) the instantaneous prices generated after accepting a vector of order $\vec{q}$ are

$$p_i = \frac{\theta_i u'(m_i)}{\sum_j \theta_j u'(m_j)} = \frac{\theta_i u'(C(\vec{q}) - q_i)}{\sum_j \theta_j u'(C(\vec{q}) - q_j))} \quad \forall i.$$

Necessarily, in this market when $\vec{q}$ orders are accepted, if $p_j > 0$, then

$$\frac{p_i}{p_j} = \frac{\theta_i u'(C(\vec{q}) - q_i)}{\theta_j u'(C(\vec{q}) - q_j)} \quad \forall i.$$

Because we showed that in the market that this utility framework imitates $\vec{p}(0) = \vec{e}/3$ and because $\sum_i \theta_i = 1$, it must be that $\vec{\theta} = \vec{e}/3$:

$$\frac{\theta_i u'(C(0) - 0)}{\theta_j u'(C(0) - 0)} = \frac{1/3}{1/3} = 1 \quad \forall i \;\Rightarrow\; \theta_i = \theta_j = 1/3 \;\forall i, \forall j.$$

However, the market should also satisfy $\vec{p}([1,1,0]^T) = [1/3, 2/3, 0]^T$ and

$$\frac{u'(C([1,1,0]^T) - 1)}{u'(C([1,1,0]^T) - 1)} = 1 \neq \frac{p_1}{p_2}.$$

This is a contradiction; thus, such a utility framework cannot exist. □

The SCPM framework therefore subsumes the set of markets that can be obtained through the expected utility framework. More importantly, because one can easily verify that in this counterexample, $v(\cdot)$ is nondecreasing and thus represents a valid risk attitude, it is actually the case that the SCPM framework can represent risk attitudes for the market maker that could not be implemented in Chen and Pennock's utility framework.

Remark 4. The relation between Chen and Pennock's framework and the SCPM is actually an intimate one, especially when we compare their respective cost function representation. Based on Theorem 7, we can first derive that the cost function associated with a utility-based market model takes the form

$$C(\vec{q}) := \underset{t}{\text{minimize}} \quad t$$

$$\text{s.t.} \quad \sum_i \theta_i u(t - q_i) \geqslant k.$$

On the other hand, one can also show that any SCPM market can be implemented using the cost function

$$C(\vec{q}) := \underset{t}{\text{minimize}} \quad t$$

$$\text{s.t.} \quad \rho(t\vec{e} - \vec{q}) \leqslant 0,$$

where $\rho(\vec{z}) = \min_t t - v(\vec{z} + t\vec{e})$ is the convex risk measure defined in Theorem 3. Notice how both cost functions charge the minimum price that is required for the market maker to preserve a constant level of risk exposure. How risk is measured in each case leads, however, to an important distinction. Because the utility framework relies on a single utility function that is replicated symmetrically for each outcome, and on subjective probabilities, it is known to have important limitations in practice; in some cases, subjective probabilities cannot be properly defined (see the famous paradox in Ellsberg 1961). The convex risk measure framework that we adopt resolves this issue, thus allowing a richer and more accurate representation of market makers.

5. Implications for Existing and New Market Mechanisms

We have shown that our unifying framework allows us to better understand the connections between some of the various market makers that have been recently introduced in the literature. However, our framework also gives many insights into how a market maker could develop a new mechanism to achieve his objectives for a market. In this section, we utilize our framework to review a nonexhaustive list of markets that can be implemented with the SCPM framework. We describe how a market maker can take insights from our development of the framework to craft SCPM markets that will satisfy his requirements for a market. Overall, we believe that these results can provide valuable guidance in designing cost-effective prediction markets.

In terms of designing and evaluating markets, there are several characteristics that must be considered. We view the primary characteristics as being:

- Truthfulness—Does the market incentivize traders to truthfully reveal their beliefs?
- Market maker's loss—Does there exist a bound on the worst-case loss for the market maker?
- Convex risk measure—Does a convex risk measure exist for the market?
- Learning—Does the market consider the benefits of learning the price distribution from the traders?

In addition to these factors, the market maker would typically be concerned about the amount of liquidity created by the market. In this work, we have not addressed that issue, but it would be an interesting topic for further study.

Table 1 summarizes conditions on SCPM and their consequences with respect to these characteristics. Next, we use these observations to evaluate various existing market mechanisms and design new mechanisms.

5.1. Evaluating Current Markets

We will initially revisit popular mechanisms in the context of their SCPM equivalent and contrast the properties that they exhibit. Detailed proofs of each property are available in Appendix A.

5.1.1. Min-SCPM Market. Our first market mechanism is the most conservative and represents the objective of minimizing the worst-case loss of the market maker. Here, value function $v(\vec{s}) = \min_i s_i$. The set of subgradients at $\vec{s} = \vec{e}$ is the convex hull of orthogonal vectors $\{\vec{e}_i\}_{i=1}^n$ where $\vec{e}_i$ denotes a vector with 1 at position i and 0 elsewhere. This convex hull is exactly the simplex. Thus, the scoring rule associated with this SCPM is proper, but not strictly proper. On the other hand, as shown in Example 2, the resulting market is the most conservative one with respect to a market maker because it guarantees him no loss. From a learning perspective, this formulation creates a constant penalty function in the dual representation. Thus, in this market there are no incentives to accept any orders from traders with beliefs different from the prior belief of the market maker.

5.1.2. Linear-SCPM Market. In some cases, the market maker may explicitly know the probabilities for the various outcomes or have very strong beliefs about these probabilities. In this situation, he may desire to employ a market mechanism that is focused on maximizing his expected return based on these beliefs. Consider a linear value function $v(\vec{s}) = \vec{\theta}^T \vec{s}$ such that $\vec{\theta} \geqslant 0, \vec{\theta}^T \vec{e} = 1$ to ensure that the SCPM model is well defined. This mechanism is not proper, because the derivative of the function is a constant vector, and does not span the simplex. Its associated cost function is $C(\vec{q}) = -\vec{\theta}^T \vec{q}$, and it does not provide any loss bound:

$$B + C(0) = \max_i \max_{\vec{s}} \{\vec{\theta}^T \vec{s} - s_i\} \geqslant \max_i \max_{s_i} (\theta_i - 1) s_i = \infty,$$

because it must be that for some i, the component θ_i is not equal to one. If $\vec{\theta}$ is positive, then it is a risk minimization market with

$$\mathscr{L}(\vec{p}) = \max_{\vec{s}} \{\vec{\theta}^T \vec{s} - \vec{p}^T \vec{s}\} = \begin{cases} 0 & \text{if } \vec{p} = \vec{\theta} \\ \infty & \text{otherwise.} \end{cases}$$

However, it is one of the most risk-taking markets. In fact, the expected loss is only bounded for the case where the true probability is exactly $\vec{\theta}$. This is because this SCPM accepts orders purely in terms of expected returns with respect to the distribution described by $\vec{\theta}$.

5.1.3. Log-SCPM Market. For the Log-SCPM presented in Peters et al. (2007), the value function is $v(\vec{s}) = \sum_i \beta_i \log s_i$, and $\nabla v(\vec{s})_i = \beta_i / s_i$, which clearly spans the interior of the simplex for any positive $\vec{\beta}$. Also, $v(\cdot)$ is smooth; thus, this mechanism is strictly proper. As presented in Example 7, $\mathscr{L}(\vec{p})$ is measured in terms of likelihood distance, which causes the worst-case loss to be unbounded.

5.1.4. Exponential-SCPM Market. In an exponential-SCPM market, the market maker characterizes his future value for surplus shares $\vec{s}$ with function $v(\vec{s}) = b \sum_i \theta_i (1 - e^{-s_i/b})$. One can show that the exponential-SCPM is associated to the cost function, $C(\vec{q}) = b \log(\sum_i \theta_i e^{q_i/b})$. The exponential-SCPM market is actually equivalent to the original logarithmic market scoring rule (LMSR) proposed by Hanson (2003), even though the cost function associated with exponential-SCPM is not the usual one that is

Table 1. A summary of conditions on SCPM and their consequences.

Condition on SCPM		Consequence
VCG pricing	$\Rightarrow$	Truthfulness, cost-function equivalence, scoring rule equivalence
VCG pricing $+ \nabla v(s)$ spans the simplex	$\Rightarrow$	Properness
VCG pricing $+ \nabla v(s)$ spans the simplex and smooth	$\Rightarrow$	Strict properness
$v(s)$ nondecreasing	$\Rightarrow$	Risk minimization equivalence

presented for LMSR (i.e., $C(\vec{q}) = b\log(\sum_i e^{q_i/b})$). By simple algebra, one can show that although our more general cost function can potentially charge a different price for an order, both functions do satisfy the conditions given by Chen and Pennock (2007) for a cost function to represent an LMSR market (see Equation (1)).[2]

As derived in Hanson (2003), we can verify the strict properness of the associated scoring rule and the worst-case loss bound, which takes the form $b\log(1/\min_i\{\theta_i\})$. However, in Hanson (2003) the authors did not know that in an exponential-SCPM or LMSR market, the market maker actually measured risks using Kullback-Leibler divergence from a prior belief $\vec{\theta}$: $\mathscr{L}(\vec{p}) = \mathscr{L}_{\text{KL}}(\vec{p}\|\vec{\theta})$. In the LMSR market based on the typical cost function mentioned earlier, the market maker assumes a uniform prior.

Remark 5. Note that different utilities can lead to equivalent markets. In particular, for any concave value function $v(\vec{s})$ that is associated to a cost function $C(\vec{q})$ as defined in Equation (4), one can instead construct an SCPM market with $\tilde{u}(\vec{s}) = -C(-\vec{s})$, which is associated to the same cost function because $C(\vec{q}) = \min_t\{t - \tilde{u}(t\vec{e} - \vec{q})\} = \min_t\{t + C(\vec{q} - t\vec{e})\} = C(\vec{q})$. This is, for example, the case for the value functions $v(\vec{s}) = -\log((1/N)\sum_i \exp(-s_i))$ and $v(\vec{s}) = \sum_i (1/N)(1-\exp(-s_i))$, which both share the cost function $C(\vec{q}) = \log(\sum_i (1/N)\exp(q_i))$. In light of this, when one designs a market, the value function $v(\vec{s})$ is easier to choose than picking a cost function that is convex and satisfies the no arbitrage condition $C(\vec{q} + r\vec{e}) = C(\vec{q}) + r$.

5.1.5. Quadratic Scoring Rule Market. Another common scoring rule is the quadratic scoring rule. In a similar manner to the LMSR, a market mechanism can be developed using the quadratic scoring rule as its foundation. A market maker using the quadratic scoring rule is equivalent to the SCPM framework with value function $v(\vec{s}) = -C(-\vec{s}) = \vec{e}^T\vec{s}/N - (1/4b)\vec{s}^T(I - (\vec{e}\vec{e}^T/N))\vec{s}$. Our results confirm that this scoring rule is strictly proper, as demonstrated in Chen and Pennock (2007), and that the worst-case loss is $(b(N-1))/N$. Unfortunately, this market does not have a valid risk interpretation and can actually lead to a negative price vector. In practice, it is typical to implement a perturbed version of this rule, which restricts the price that is charged to be positive. Unfortunately, no analysis has yet been done on the consequence of using this perturbed quadratic scoring rule market.

Thus, we are able to describe the key characteristics of these mechanisms by using insights from our framework. Table 2 summarizes the conclusions we derived for these mechanisms.

5.2. New Market Design

In this section, we demonstrate applications of our unified tool for designing new market mechanisms with desirable properties. Because all the properties of a mechanism in our framework are characterized by the properties of the value function, the mechanism design problem reduces to picking an appropriate concave value function. As a result, we design the Quad-SCPM mechanism, a much-improved version of the existing quadratic scoring rule, and P-linear-SCPM, an improvement over linear-SCPM. Both are obtained by slightly modifying the value function used in the original markets to remove their negative traits. The generalization of LMSR to Exp-SCPM discussed in the last section is another example where our framework was used to improve and extend popular mechanisms. Quad-SCPM is a particularly interesting new market because it simultaneously achieves the properties of nonnegative prices, strict properness, bounded loss (independent of the number of states), and intuitive risk measure. None of the existing mechanisms provided all of these properties simultaneously.

As another approach for new market design, we will focus on the development of market mechanisms that emphasize the learning of specific distribution information from the trading population. This will use the theory developed in §3.4 to capture the risk attitude of the market maker using specific value functions.

Table 2. A summary of properties of various market mechanisms (when $\vec{\theta} = (1/N)\vec{e}$).

	Truthful	Convex risk Worst cost	Measure	$L(\vec{p})$	Properness
Min-SCPM	Yes	0	Yes	0	Proper
Linear-SCPM	Yes	∞	Yes	0 or ∞	Not proper
Log-SCPM	Yes	∞	Yes	$b\mathscr{L}_{\text{LL}}(\vec{p}\|U)$	Strictly proper
LMSR	Yes	$b\log N$	Yes	$b\mathscr{L}_{\text{KL}}(\vec{p}\|U)$	Strictly proper
Exponential-SCPM	Yes	$b\log N$	Yes	$b\mathscr{L}_{\text{KL}}(\vec{p}\|U)$	Strictly proper
Quad. scoring rule	Yes	$b\dfrac{N-1}{N}$	No	—	Strictly proper
Quad-SCPM	Yes	$b\dfrac{N-1}{N}$	Yes	$b\|\vec{p}-U\|_2^2$	Strictly proper
P-Linear-SCPM	Yes	$b\dfrac{N-1}{N}$	Yes	$\dfrac{b}{2}\|\vec{p}-U\|_1$	Proper

5.2.1. Quad-SCPM Market. Our unified analysis of the SCPM model allows us to suggest a modification to the prediction market that uses a quadratic scoring rule. Although this rule is known to be myopically truthful, in practice a market maker that uses this rule needs to explicitly restrict prices to be between 0 and 1 at all times. A solution to this problem is to use an SCPM market with the following semiquadratic nondecreasing value function. Consider the "quad-SCPM" obtained from using the following value function:

$$v(\vec{s}) = \max_{\vec{d}\leqslant\vec{s}} \vec{\theta}^T\vec{d} - \frac{1}{4b}\vec{d}^T\vec{d}$$

for some $\vec{\theta}$ such that $\vec{\theta}\geqslant 0$, $\sum_i \theta_i = 1$. This value function is nondecreasing and concave, which ensures that resulting prices are nonnegative. It has bounded worst-case loss given by $b(\|\vec{\theta}\|_2^2 + 1 - 2\min_i \theta_i)$. The distance from the prior $\hat{p} = \vec{\theta}$ is measured by $\mathcal{L}(\vec{p}) = b\|\vec{p}-\vec{\theta}\|_2^2$, which is the 2-norm distance. The resulting prediction market is myopically truthful and leads to orders that can be priced using the cost function:

$$C(\vec{q}) = \min_{t,\,\vec{d}\leqslant t\vec{e}-\vec{q}} t - \vec{\theta}^T\vec{d} + \frac{1}{4b}\vec{d}^T\vec{d},$$

which requires solving a quadratic program. Also, the corresponding scoring rule is strictly proper.

One can verify that the above market is closely related to the quadratic scoring rule market, because $C(\vec{q})$ reduces to the quadratic scoring rule cost function when $\vec{\theta} = \vec{e}/N$, and the constraint $\vec{d}\leqslant t\vec{e}-\vec{q}$ is replaced with $\vec{d} = t\vec{e}-\vec{q}$. However, the quad-SCPM market always has positive prices and has an intuitive interpretation in terms of distance to the prior belief. Because the worst-case loss is actually naturally bounded by a value b that does not depend on the size of the outcome space, the quad-SCPM seems to be the perfect choice for markets with infinite outcome space.

5.2.2. P-Linear-SCPM Market. Next, we consider an improved version of linear-SCPM market discussed earlier. We show that a small modification to the value function can handle the problem of unbounded loss in this market. We call the new market P-linear-SCPM, which stands for piecewise-linear SCPM. The "P-linear-SCPM" is obtained by using the following piecewise-linear value function:

$$v(\vec{s}) = \begin{cases} \sum_i \theta_i \min(0, s_i) & \text{if } \vec{s}\geqslant -b\vec{e} \\ -\infty & \text{otherwise.} \end{cases}$$

for some $\vec{\theta}$ such that $\vec{\theta}\geqslant 0$, $\sum_i \theta_i = 1$. This value function is concave, nondecreasing and separable. It has a bounded worst-case loss equal to $b(1-\min_i \theta_i)$. The market is known to be myopically truthful when the orders are priced using the convex cost function

$$C(\vec{q}) = \min_{t\vec{e}\geqslant\vec{q}-b\vec{e}} \left\{ t - \sum_i \theta_i \min(0, t-q_i) \right\},$$

which only requires solving a simple linear program. An interesting property of this market is that the risk interpretation is characterized using the "total variation" distance from the prior $\vec{\theta}$: $\mathcal{L}(\vec{p}) = (b/2)\|\vec{p}-\vec{\theta}\|_1$. In contrast to linear SCPM, the P-linear mechanism is proper (although not strictly proper).

5.2.3. Distribution Information Learning Markets. In this section, we focus on the development of new market mechanisms that emphasize the learning of specific distribution information from the trading population. Typically, markets have been designed to learn the probability that each outcome occurs. When the outcome space becomes large, this cannot be achieved anymore. Instead, one needs to deal with the fact that he can only learn a subset of information about the distribution. In this case, the theory developed in §3.4 can be used to define a value function

$$v(\vec{s}) = \min_{\vec{p}\geqslant 0}\{\vec{p}^T\vec{s} + \mathcal{L}(\vec{p})\}$$

for some well-designed convex penalty function $\mathcal{L}(\vec{p})$ that encodes one's learning priorities.

For instance, a priority of the market might be to learn the mean of a random vector X defined over the space of outcomes. It is actually the case that an SCPM market that uses a value function constructed from the penalty function $\mathcal{L}(p) = b\|\sum_i p_i\vec{X}(i) - \vec{\mu}\|^2$ will achieve this objective. Intuitively, such an SCPM market invests in learning that the mean of the random vector $\vec{X}$ is not close to an estimate $\vec{\mu}$. Actually, the quad-SCPM market is a simple case of this approach with $\vec{X}(i)\in\Re^N$ such that $X_j(i) = 1$ if $i = j$ and 0 otherwise, which learns the probability distribution of a discrete set of outcomes. In the case that the outcome space is large, $i\in\{1, 2, \ldots, N\}$, one can consider $\vec{X}\in\Re^K$, for some $K \ll N$, such that

$$\vec{X}_j(i) = \begin{cases} 1 & \text{if } i\in\mathcal{P}_j \\ 0 & \text{otherwise,} \end{cases}$$

where $\{\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_K\}$ is a partition of $\{1, 2, \ldots, N\}$. The resulting SCPM market will learn the probability that the outcome falls in each of the K partitions and has worst-case loss again bounded by b. In theory, this approach can even be extended to an infinite outcome space and more complex definitions for $\vec{X}$ with the guarantees that worst-case loss is bounded by $2b$ times the square of the largest achievable norm for $\vec{X}$. In practice, however, one needs to also make sure that the resulting cost function can be evaluated (or approximated) efficiently for the type of orders that are submitted.

6. Conclusion

In this work, we introduced a unified convex optimization framework for constructing prediction market mechanisms. We first showed that in this new framework, the pricing mechanism always allows truthful orders (in a myopic sense) to be a strategy that is optimal for the traders. Also, the pricing mechanism can be computed efficiently using a convenient convex cost function formulation. We showed how markets using scoring rules or Chen and Pennock's utility framework could be cast in this rich unifying framework. These popular mechanisms and the original Log-SCPM mechanism actually differ only in terms of the choice of a value function for the generalized SCPM framework. We also showed how to analyze and compare the properties of the markets obtained using any given value functions: worst-case loss, risk attitude, and properness as defined for scoring rules. Table 1 summarized conditions on SCPM and their consequences with respect to these properties. Table 2 listed the conclusions we derived for various mechanisms. We believe these properties are of particular interest when designing a prediction market. In fact, our study allowed us to derive the quad-SCPM market, which is the first mechanism to implement properness, truthfulness, risk control, and a constant loss bound simultaneously. The results presented in this work also allow the possibility of designing markets according to what the market maker wishes to achieve in terms of learning and according to how much he is willing to invest for this information.

We believe that our framework for designing dynamic prediction markets has intimate connections to other dynamic trading markets such as online auction of goods, and could lead to interesting results for these markets as well. In general, any dynamic resource allocation and pricing scheme relies crucially on the trade-off between the profit achieved by exploiting the resource now versus the value of saving the goods for the future and exploring the market further. This future value of resources is captured in our framework by the concave value function $v(\cdot)$. Our risk-based formulation also formalized how this value function captures the trade-off between learning the preferences of the traders versus maximizing instant profit via a penalty function. This bears similarities to the classic exploration versus exploitation trade-off for general trading markets. Additionally, our mechanism achieves incentive compatibility using the VCG allocation and pricing scheme popular for online auctions of goods. Further investigation of implications of our results on other trading and auction markets is part of an ongoing research.

Appendix A. Properties of Various SCPM Mechanisms

A.1. Properties of the Min-SCPM

• *Worst-Case Loss*: Necessarily,

$$C(0) = \min_t \{t - \min_i t_i\} = 0, \qquad \text{whereas}$$

$$B = \max_i \max_{\vec{s}} \{\min_j s_j - s_i\} = 0.$$

Therefore, the worst-case loss is 0.

• *Risk Attitude*: The penalty function can be derived as follows:

$$\begin{aligned}\mathcal{L}(\vec{p}) &= \max_{\vec{s}} \left\{ \min_i s_i - \vec{p}^T \vec{s} \right\} \\ &= \max_{\vec{s}t \leqslant s_i, \forall i} \{t - \vec{p}^T \vec{s}\} \\ &= \max_{\vec{s}, t} \min_{\lambda} \{t - \vec{p}^T \vec{s} + \lambda^T (\vec{s} - t\vec{e})\} \\ &= \min_{\lambda} \max_{\vec{s}, t} \{t - \vec{p}^T \vec{s} + \lambda^T (\vec{s} - t\vec{e})\} = 0.\end{aligned}$$

• *Properness*: The set of subgradients at $\vec{s} = \vec{e}$ is the convex hull of orthogonal vectors $\{\vec{e}_i\}_{i=1}^n$ where $\vec{e}_i$ denotes a vector with 1 at position i and 0 elsewhere. This convex hull is exactly the simplex. Thus, the scoring rule associated with this SCPM is proper, but not strictly proper.

A.2. Properties of the Log-SCPM

• *Worst-Case Loss*: The worst-case loss is unbounded because $C(0) = (\sum_i \beta_i)(1 - \log \sum_i \beta_i)$ and $B = \infty$. Specifically,

$$C(0) = \min_t \left\{ t - \sum_i \beta_i \log(t) \right\} = \left(\sum_i \beta_i \right) \left(1 - \log \sum_i \beta_i \right)$$

and

$$\begin{aligned}B = \max_i \max_{\vec{s}} \left\{ \sum_j \beta_j \log(s_j) - s_i \right\} &\geqslant \lim_{\alpha \to \infty} \beta_1 \log(1) \\ &+ \sum_{j \neq 1} \beta_j \log(\alpha) - 1 = \infty,\end{aligned}$$

where we restricted the optimization to $s_1 = 1$ and $s_j = \alpha$, $\forall j \neq 1$ and assumed without loss of generality that $\beta_j \neq 0$ for some $j \neq 1$.

• *Risk Attitude*: The penalty function $\mathcal{L}(\vec{p})$ can be derived by simple algebra. Given that $v(\cdot)$ has the form $v(\vec{s}) = \sum_i \beta_i \log(s_i)$, we can show that

$$\begin{aligned}\mathcal{L}(\vec{p}) &= \max_{\vec{s}} \left\{ \sum_i \beta_i \log(s_i) - p_i s_i \right\} \\ &= \sum_i (\beta_i \log(\beta_i / p_i) - \beta_i) \\ &= -\sum_i \beta_i \log(p_i) + \sum_i (\beta_i \log \beta_i - \beta_i) \\ &= \left(\sum_i \beta_i \right) \mathcal{L}_{\mathrm{LL}}(\vec{p} \| \vec{\theta}) + k,\end{aligned}$$

where $\theta_i = \beta_i / \sum_i \beta_i$, and $\mathcal{L}_{\mathrm{LL}}(\vec{p} \| \vec{\theta}) = -\log(\prod_i p_i^{\theta_i})$. As $\sum_i \beta_i$ increases the curvature of the negative log-likelihood function increases. Thus, a higher $\sum_i \beta_i$ leads to more risk tolerance.

• *Properness*: Here, $\nabla v(\vec{s})_i = \beta_i / s_i$, which clearly spans the interior of the simplex for any positive $\vec{\beta}$. Also, $v(\cdot)$ is smooth; thus, this mechanism is strictly proper.

A.3. Properties of the Exponential-SCPM

- *Cost function*:

$$C(\vec{q}) = \min_t \left\{ t - b\sum_i \theta_i (1 - e^{(-t+q_i)/b}) \right\}$$
$$= \min_t \left\{ t + be^{-t/b} \sum_i \theta_i e^{q_i/b} - b\sum_i \theta_i \right\}.$$

Above is minimized at

$$\sum_i \theta_i e^{q_i/b} e^{-t/b} = 1 \ \Rightarrow\ t^* = b\log\left(\sum_i \theta_i e^{q_i/b}\right).$$

Thus,

$$C(\vec{q}) = b\log\left(\sum_i \theta_i e^{q_i/b}\right).$$

- *Worst-Case Loss*: First, we can verify that $C(0) = 0$. We then need to resolve that

$$B = \max_i \max_{\vec{s}} \left\{ b\sum_j \theta_j (1 - e^{-s_j/b}) - s_i \right\}$$
$$= \max_i \max_{\vec{s}} \left\{ b - b\sum_j \theta_j e^{-s_j/b} - s_i \right\}$$
$$= \max_i \max_{s_i} \left\{ b - b\theta_i e^{-s_i/b} - s_i \right\} = \max_i \{ b\log(1/\theta_i) \}.$$

Therefore, the worst-case loss is $b\log(1/\min_i\{\theta_i\})$.

- *Risk Attitude*: We simply resolve the definition of $\mathscr{L}(\vec{p})$.

$$\mathscr{L}(\vec{p}) = \max_{\vec{s}} \left\{ b\sum_i \theta_i (1 - e^{-s_i/b}) - \vec{p}^T\vec{s} \right\}$$
$$= b - \sum_i \min_{s_i} \{ b\theta_i e^{-s_i/b} + p_i s_i \}$$
$$= -b\sum_i p_i \log(\theta_i/p_i) = b\mathscr{L}_{\mathrm{KL}}(\vec{p}\,\|\,\vec{\theta}).$$

- *Properness*: The function $v(\cdot)$ is smooth, and the gradient $\nabla v(\vec{s})_i = \theta_i e^{-s_i/b}$. Clearly, the gradient spans the simplex as long as $\theta_i > 0$ for all i, thus, this is the only condition for this mechanism to be strictly proper.

A.4. Properties of the LMSR

Because the LMSR generates the same market as the exponential-SCPM, we refer the reader to §A.3, where we derive all the properties of the exponential-SCPM.

A.5. Properties of the Quadratic Scoring Rule

- *Worst-Case Loss*: When $v(\vec{s}) = \vec{e}^T\vec{s}/N - (1/4b)\vec{s}^T P\vec{s}$ with $P = (I - (1/N)\vec{e}\vec{e}^T)$, we first show that $C(0) = 0$, and then that $B = b(1 - (1/N))$:

$$C(0) = \min_t \{ t - v(t\vec{e}) \} = \min_t \left\{ t + \frac{1}{4b} t^2 \vec{e}^T P\vec{e} - t \right\} = 0$$

and

$$B = \max_i \max_{\vec{s}} \{ v(\vec{s}) - s_i \} = v([-2b, 0, \ldots, 0]^T) + 2b$$
$$= \frac{-2b}{N} - b\left(1 - \frac{1}{N}\right) + 2b = b\left(1 - \frac{1}{N}\right).$$

- *Risk Attitude*: Because the derivative of $v(\vec{s}) = (1/N)\vec{e}^T\vec{s} - (1/4b)\vec{s}^T P\vec{s}$ is not nondecreasing, this version of the SCPM does not have an equivalent representation in terms of convex risk minimization. Specifically, when $\vec{s} = (2b, 0, 0, \ldots, 0)$, we can verify that $\partial v(\vec{s}))/\partial s_1 = -1 + (2/N) < 0$.
- *Properness*: Our condition for strict properness is satisfied because $v(\cdot)$ is smooth and

$$\nabla v(\vec{s}) = \left[\frac{1}{N} + \frac{(\vec{e}^T\vec{s}/N) - s_i}{2b} \right].$$

Thus, for any vector $\vec{r}$ in the simplex, we can set $s_i = -2br_i$ to get $\nabla v(\vec{s}) = \vec{r}$.

A.6. Properties of the Quad-SCPM

- *Cost function*: The cost function for this model can only be expressed in its optimization form: $C(\vec{q}) = \min_{t,\vec{w}\leqslant t\vec{e}-\vec{q}} \{ t - \vec{\theta}^T\vec{w} - (1/4b)\vec{w}^T\vec{w} \}$.
- *Worst-Case Loss*: First, one can show that $C(0) = 0$:

$$C(0) = \min_t \left\{ t - \max_{\vec{w}\leqslant t\vec{e}} \left\{ \vec{\theta}^T\vec{w} - \frac{1}{4b}\vec{w}^T\vec{w} \right\} \right\}$$
$$= \min_{t,\vec{w}} \max_{\vec{p}\geqslant 0} \left\{ t - \vec{\theta}^T\vec{w} + \frac{1}{4b}\vec{w}^T\vec{w} - \vec{p}^T(t\vec{e} - \vec{w}) \right\}$$
$$= \max_{\vec{p}\geqslant 0} \min_{t,\vec{w}} \left\{ t - \vec{\theta}^T\vec{w} + \frac{1}{4b}\vec{w}^T\vec{w} - \vec{p}^T(t\vec{e} - \vec{w}) \right\}$$
$$= \max_{\vec{p}\geqslant 0,\, \vec{e}^T\vec{p}=1} \{ -b\|\vec{\theta} - \vec{p}\|_2^2 \} = 0.$$

To compute B, we have

$$\max_i \max_{\vec{s}} \{ v(\vec{s}) - s_i \}$$
$$= \max_i \max_{\vec{s},\vec{w}\leqslant\vec{s}} \left\{ \vec{\theta}^T\vec{w} - \frac{1}{4b}\vec{w}^T\vec{w} - s_i \right\}$$
$$= \max_i \max_{\vec{s},\vec{w}} \min_{\vec{p}\geqslant 0} \left\{ \vec{\theta}^T\vec{w} - \frac{1}{4b}\vec{w}^T\vec{w} - s_i + \vec{p}^T(\vec{s} - \vec{w}) \right\}$$
$$= \max_i \min_{\vec{p}\geqslant 0} \max_{\vec{s},\vec{w}} \left\{ \vec{\theta}^T\vec{w} - \frac{1}{4b}\vec{w}^T\vec{w} - s_i + \vec{p}^T(\vec{s} - \vec{w}) \right\}$$
$$= \max_i \{ b((\theta_i - 1)^2 + \|\vec{\theta}\|_2^2 - \theta_i^2) \}$$
$$= \max_i \{ b(-2\theta_i + 1 + \|\vec{\theta}\|_2^2) \}$$
$$= 2b\left(\tfrac{1}{2} + \tfrac{1}{2}\|\vec{\theta}\|_2^2 - \min_i \theta_i\right).$$

Thus, the worst-case loss is $2b(\frac{1}{2} + \frac{1}{2}\|\vec{\theta}\|_2^2 - \min_i \theta_i)$, which is actually never greater than b because, by definition, $\vec{\theta}$ lies in the simplex.

- *Risk Attitude*: We simply resolve the definition of $\mathscr{L}(\vec{p})$:

$$\begin{aligned}\mathscr{L}(\vec{p}) &= \max_{\vec{s},\,\vec{w}\leqslant\vec{s}}\left\{\vec{\theta}^T\vec{w}-\frac{1}{4b}\vec{w}^T\vec{w}-\vec{p}^T\vec{s}\right\}\\ &= \max_{\vec{s},\,\vec{w}}\min_{\lambda\geqslant 0}\left\{\vec{\theta}^T\vec{w}-\frac{1}{4b}\vec{w}^T\vec{w}-\vec{p}^T\vec{s}+\lambda^T(\vec{s}-\vec{w})\right\}\\ &= b\|\vec{p}-\vec{\theta}\|_2^2.\end{aligned}$$

- *Properness*: The value function $v(\vec{s})$ at a given $\vec{s}\geqslant -b\vec{e}$ is given by

$$\begin{aligned}\underset{\vec{d}}{\text{maximize}}\quad & \frac{1}{N}\vec{\theta}^T\vec{d}-\frac{1}{4b}\vec{d}^T\vec{d}\\ \text{s.t.}\quad & \vec{d}\leqslant\vec{s}.\end{aligned}$$

The partial derivative of this function with respect to s_i is 0 at $s_i\geqslant 2\theta_i b/N$, and $\theta_i/N - s_i/2b$ at $s_i < 2\theta_i b/N$. Therefore, gradient $\nabla v(\vec{s})_i = \max\{0, \theta_i/N - s_i/2b\}$, which spans the simplex and is continuous on the simplex. Thus, this mechanism is strictly proper.

A.7. Properties of the P-Linear-SCPM

- *Cost function*: The cost function for this model can only be expressed in its optimization form: $C(\vec{q}) = \min_{t,\,t\vec{e}-\vec{q}\geqslant b}\{t-\sum_i\theta_i\min(0, t-q_i)\}$.
- *Worst-case loss*: The worst-case loss is $\max_i b(1-\theta_i)$ because one can show that

$$C(0)=\min_{t\geqslant -b}\left\{t-\sum_i\theta_i\min(0,t)\right\}=0,$$

and then compute B as follows:

$$\begin{aligned}\max_{i,\,s}\{v(\vec{s})-s_i\} &= \max_{i,s\geqslant -be}\left\{\sum_j\theta_j\min(0,s_j)-s_i\right\}\\ &= \max_i\left\{\max_{s_i\geqslant -b}\theta_i\min(0,s_i)-s_i\right\}\\ &= \max_i\{b(1-\theta_i)\}.\end{aligned}$$

- *Risk Attitude*: We simply resolve the definition of $\mathscr{L}(\vec{p})$.

$$\begin{aligned}\mathscr{L}(\vec{p}) &= \max_{\vec{s}\geqslant -b\vec{e}}\left\{\sum_i\theta_i\min\{0,s_i\}-p^Ts\right\}\\ &= \sum_i\max\left\{\max_{-b\leqslant s_i\leqslant 0}(\theta_i-p_i)s_i,\ \max_{s\geqslant 0}-p_is_i\right\}\\ &= \sum_i\max\{\max\{0, b(p_i-\theta_i)\}0\}\\ &= \frac{b}{2}\|\vec{p}-\vec{\theta}\|_1.\end{aligned}$$

- *Properness*: Consider $\vec{s}=0$. The set of subgradients of $v(0)$ is the convex hull of the set $\{0,1\}^n$. Therefore, the set of subgradients spans the simplex. However, the price vector is not unique, so the mechanism is proper, but not strictly proper.

Appendix B. Convexity of Risk Measure $\rho(\vec{Z})$

- *Convexity*: Because $\rho(Z)=\min_t\{t-v(t\vec{e}+\vec{Z})\}$, and $v(\cdot)$ is concave, we know that $\rho(Z)$ is convex.
- *Monotonicity*: The monotonicity also simply results from the monotonicity of $v(\cdot)$. Given that $\vec{Z}\succeq\vec{Z}'$, then

$$\rho(Z)=\min_t t-v(\vec{Z}+t\vec{e})\leqslant\min_t t-v(\vec{Z}'+t\vec{e})=\rho(Z'),$$

because the inequality is true for any fixed value of t.

- *Translation equivariance*: Finally, translation equivariance can be simply demonstrated with a change of variable $t'=t+\alpha$:

$$\begin{aligned}\rho(Z+\alpha) &= \min_t t-v(\vec{Z}+(\alpha+t)\vec{e})\\ &= \min_{t'} t'-\alpha-v(\vec{Z}+t')\\ &= \rho(Z)-\alpha.\end{aligned}$$

Endnotes

1. There is another technical condition on $v(\cdot)$ that is required for this model to be feasible and bounded. The condition is that $\forall\vec{q}\geqslant 0$, $\exists t$, $\nabla v(t\vec{e}-\vec{q})^T\vec{e}=1$, where $\nabla v(\cdot)$ denotes the (sub)gradient function.
2. First, one can establish the following relation between the two cost functions:

$$\begin{aligned}C_1(\vec{q}) &= b\log\left(\sum_i e^{q_i/b}\right)\\ C_2(\vec{q}) &= b\log\left(\sum_i\theta_i e^{q_i/b}\right)=b\log\left(\sum_i\frac{e^{r_i/b}}{\sum_i e^{r_i/b}}e^{q_i/b}\right)\\ &= b\log\left(\sum_i e^{(q_i+r_i)/b}\right)-b\log\left(\sum_i e^{r_i/b}\right)\\ &= C_1(\vec{q}+\vec{r})-C_1(\vec{r}),\end{aligned}$$

where $\vec{r}$ is such that $\vec{\theta}_i = e^{r_i/b}/\sum_i e^{r_i/b}$. In general, this relation ensures that the two cost functions can implement the same scoring rule. Specifically, if $C_1(\cdot)$ satisfies the condition layed out by Chen and Pennock for a cost function to represent scroring rule $\mathscr{S}$, then so does $C_2(\cdot)$. First, the relation between $\mathscr{S}$ and $C_1(\cdot)$ implies that for all $\vec{p}$ in the probability simplex, there exists a vector $\vec{q}$ such that $\mathscr{S}_i(\vec{p})=q_i-C_1(\vec{q})$ and $p_i=\partial C_1/\partial q_i|_{\vec{q}}$. However, it is also the case that $p_i=\partial C_2/\partial q_i|_{\vec{q}'=\vec{q}-\vec{r}}$. Thus, $C_2(\cdot)$ is the cost function for a scoring rule $\mathscr{S}'$ that is equivalent to $\mathscr{S}$:

$$\mathscr{S}'_i(\vec{p})=q_i-r_i-C_2(\vec{q}-\vec{r})=q_i-C_1(\vec{q})+C_1(\vec{r})-r_i.$$

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