Midterm Review

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(LY, Chapter 1- 6, Appendices, and Lecture Notes 1-8)

Separating hyperplane theorem

The most important theorem about the convex set is the following separating theorem.

Theorem 1 *(Separating hyperplane theorem) Let* $C \subset \mathcal{E}$*, where* $\mathcal{E}$ *is a metric space either* $\mathcal{R}^n$ *or* $\mathcal{M}^n$*, be a* closed *convex set and let* $\mathbf{y}$ *be a point exterior to* C*. Then there is a point* $\mathbf{a} \in \mathcal{E}$ *such that*

$$\mathbf{a} \bullet \mathbf{y} > \sup_{\mathbf{x} \in C} \mathbf{a} \bullet \mathbf{x}.$$

Carathéodory's theorem

Theorem 2 *Given matrix $A \in \mathcal{R}^{m\times n}$ and $\mathbf{b} \in \mathcal{R}^m$. If*

$$\{\mathbf{x}: \ A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$$

is feasible. Then

$x_B = A_B^{-1} b$

$$\mathbf{b} = A_B\mathbf{x}_B, \ \mathbf{x}_B \geq \mathbf{0}$$

where columns in A_B are linearly independent chosen from $\mathbf{a}_1$,...,$\mathbf{a}_n$ of A.

The theorem is true whether or not A has a full row rank or not.

$\boxed{\text{S-theorem}}$ $\boxed{\text{C-theorem}}$

Farkas' Lemma

The following results are Farkas' lemma and its variants.

Theorem 3 *Let* $A \in \mathcal{R}^{m \times n}$ *and* $\mathbf{b} \in \mathcal{R}^m$. *Then, the system* $\{\mathbf{x} : \; A\mathbf{x} = \mathbf{b}, \; \mathbf{x} \geq \mathbf{0}\}$ *has a feasible solution* $\mathbf{x}$ *if and only if that system* $\{\mathbf{y} : \; A^T\mathbf{y} \leq \mathbf{0}, \; \mathbf{b}^T\mathbf{y} > 0\}$ *has no feasible solution* $\mathbf{y}$.

$\in R^n_+$ $\in R^m_+$

A vector $\mathbf{y}$, with $A^T\mathbf{y} \leq \mathbf{0}$ and $\mathbf{b}^T\mathbf{y} > 0$, is called an infeasibility certificate for the system $\{x : \; Ax = b, \; x \geq 0\}$.

Geometrically, Farkas' lemma means that if a vector $\mathbf{b} \in \mathcal{R}^m$ does not belong to the cone generated by columns of A, then there is a hyperplane separating $\mathbf{b}$ from the cone $\text{cone}(\mathbf{a}_{.1}, ..., \mathbf{a}_{.n})$.

Example

Let $A = (1, 1)$ and $b = -1$. Then, $y = -1$ is an infeasibility certificate for $\{\mathbf{x} : \; A\mathbf{x} = b, \; \mathbf{x} \geq \mathbf{0}\}$.

Farkas' Lemma Variant

Theorem 4 *Let* $A \in \mathcal{R}^{m\times n}$ *and* $\mathbf{c} \in \mathcal{R}^n$*. Then, the system* $\{\mathbf{y} : \; A^T\mathbf{y} \le \mathbf{c}\}$ *has a solution* $\mathbf{y}$ *if and only if that system* $\{\mathbf{x} : \; A\mathbf{x} = \mathbf{0}, \; \mathbf{c}^T\mathbf{x} < 0, \; \mathbf{x} \ge \mathbf{0}\}$ *has no feasible solution* $\mathbf{x}$*.*

Again, a vector $\mathbf{x} \ge \mathbf{0}$, with $A\mathbf{x} = \mathbf{0}$ and $\mathbf{c}^T\mathbf{x} < 0$, is called a infeasibility certificate for the system $\{\mathbf{y} : \; A^T\mathbf{y} \le \mathbf{c}\}$.

example

Let $A = (1; -1)$ and $\mathbf{c} = (1; -2)$. Then, $\mathbf{x} = (1; 1)$ is an infeasibility certificate for $\{y : \; A^T y \le \mathbf{c}\}$.

Farkas' Lemma for General Cones?

Given $\mathbf{a}_i$, $i = 1, ..., m$, and $\mathbf{b} \in \mathcal{R}^m$. Convex & closed

Then, the system $\{\mathbf{x} : \ \mathbf{a}_i \bullet \mathbf{x} = b_i, \ i = 1, ..., m, \ \mathbf{x} \in C\}$ has a feasible solution $\mathbf{x}$ if and only if that $-\sum_i^m y_i \mathbf{a}_i \in C^*$ and $\mathbf{b}^T\mathbf{y} > 0$ has no feasible solution $\mathbf{y}$?

It is necessary but not sufficient!

LP & SDP

Let's write equations in a compact form:

$$\mathcal{A}\mathbf{x} = (\mathbf{a}_1 \bullet \mathbf{x}; ...; \mathbf{a}_m \bullet \mathbf{x}) \in \mathcal{R}^m$$

and

$$\mathcal{A}^T\mathbf{y} = \sum_i^m y_i \mathbf{a}_i.$$

Alternative Systems for General Cones?

Alternative System Pair I?:

$$\mathcal{A}\mathbf{x} = \mathbf{b}, \quad \mathbf{x} \in C,$$

and

$$-\mathcal{A}^T\mathbf{y} \in C^*, \quad \mathbf{b}^T\mathbf{y} \overset{(>1)}{=} 1 \quad (b^T y > 0)$$

Alternative System Pair II?:

$$\mathcal{A}\mathbf{x} = \mathbf{0}, \quad \mathbf{x} \in C, \quad \mathbf{c} \bullet \mathbf{x} = -1 (< 0)$$

and

$$\mathbf{c} - \mathcal{A}^T\mathbf{y} \in C^*$$

When Farkas' Lemma Holds for General Cones?

Let C be a closed convex cone in the rest of the course.

If there is $\bar{\mathbf{y}}$ such that $-A^T\bar{\mathbf{y}} \in \text{int}\, C^*$, then Alternative System Pair I is true:

$$A\mathbf{x} = \mathbf{b}, \quad \mathbf{x} \in C,$$

and

$$-A^T\mathbf{y} \in C^*, \quad \mathbf{b}^T\mathbf{y} = 1$$

$\{Ax,\ x \in C\}$

$-\bar{y}^T b = Ax$

$= -\bar{y}^T Ax$

$= \boxed{(-\bar{y}^T A)} \cdot x$

> 0

And if there is $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{0},\ \mathbf{x} \in \text{int}\, C$, then Alternative System Pair II is true:

$$A\mathbf{x} = \mathbf{0}, \quad \mathbf{x} \in C, \quad \mathbf{c} \bullet \mathbf{x} = -1 (< 0)$$

and

$$\mathbf{c} - A^T\mathbf{y} \in C^*$$

Farkas's lemma

Duality Theory

Consider the linear program in standard form, called the primal problem,

$$\boxed{\begin{array}{llll} (LP) & \text{minimize} & \mathbf{c}^T\mathbf{x} & \\ & \text{subject to} & A\mathbf{x} = \mathbf{b}, \ \mathbf{x} \geq \mathbf{0}, & \mathcal{R}^n_+ \end{array}}$$

where decision variables $\mathbf{x} \in \mathcal{R}^n$.

The dual problem can be written as:

$$\begin{array}{lll} (LD) & \text{maximize} & \mathbf{b}^T\mathbf{y} \\ & \text{subject to} & A^T\mathbf{y} + \mathbf{s} = \mathbf{c}, \ \mathbf{s} \geq \mathbf{0}, \quad \mathcal{R}^n_+ \end{array}$$

where $\mathbf{y} \in \mathcal{R}^m$ and $\mathbf{s} \in \mathcal{R}^n$. The components of $\mathbf{s}$ are often called dual slacks.

Duality Theory

Theorem 5 *(Weak duality theorem) Let primal feasible set $\mathcal{F}_p$ and dual feasible set $\mathcal{F}_d$ be non-empty. Then,*

$$\mathbf{c}^T\mathbf{x} \ge \mathbf{b}^T\mathbf{y} \quad \textit{where} \quad \mathbf{x} \in \mathcal{F}_p,\ (\mathbf{y}, \mathbf{s}) \in \mathcal{F}_d.$$

This theorem shows that a feasible solution to either problem yields a bound on the value of the other problem. We call $\mathbf{c}^T\mathbf{x} - \mathbf{b}^T\mathbf{y}$ the duality gap.

Theorem 6 *(Strong duality theorem) Let primal feasible set $\mathcal{F}_p$ and dual feasible set $\mathcal{F}_d$ be non-empty. Then, $\mathbf{x}^* \in \mathcal{F}_p$ is optimal for (LP) and $(\mathbf{y}^*, \mathbf{s}^*) \in \mathcal{F}_d$ is optimal for (LD) if and only if*

$$\mathbf{c}^T\mathbf{x}^* = \mathbf{b}^T\mathbf{y}^*.$$

$Ax=b, x\ge 0$

$A^Ty \le c$

$c^Tx \ge b^Ty$

$\boxed{c^Tx \le b^Ty}$

Theorem 7 *(Primal-Dual relation theorem) If (LP) and (LD) both have feasible solutions then both problems have optimal solutions and the optimal objective values of the objective functions are equal.*

If one of (LP) or (LD) has no feasible solution, then the other is either unbounded or has no feasible solution either. If one of (LP) or (LD) is unbounded then the other has no feasible solution.

The above theorems show that if a pair of feasible solutions can be found to the primal and dual problems with equal objective values, then they are both optimal, respectively.

The converse is also true; there is zero duality “gap’ if they are optimal.

Duality and Complementarity Gaps

$Ax = b,\ x \geq 0$

$A^T y + s = c,\ s \geq 0$

$\mathbf{x} \in \mathcal{F}_p$ is optimal for (LP) and $(\mathbf{y}, \mathbf{s}) \in \mathcal{F}_d$ is optimal for (LD)

$$\mathbf{x}^T\mathbf{s} = \mathbf{x}^T(\mathbf{c} - A^T\mathbf{y}) = \mathbf{c}^T\mathbf{x} - \mathbf{b}^T\mathbf{y} = 0$$

implies that $x_j s_j = 0$ for all $j = 1, ..., n$, since both $\mathbf{x}$ and $\mathbf{s}$ are nonnegative. This is called the complementarity gap.

$[A\bar{x} = b,\ \bar{x} \geq 0]$ $\quad [A^T\bar{y} + \bar{s} = 0,\ \bar{s} \geq 0]$

$A(\bar{x} + \alpha u) = b$

$Ax^k = b,\ x^k \geq 0$ or $A^T y^k + s^k = 0,\ s^k \geq 0$

≥ 0

such that $\|x^k\| \to \infty$ or $\|s^k\| \to \infty$

$$\begin{aligned} x_j s_j &= 0,\ \forall j \\ A\mathbf{x} &= \mathbf{b} \\ -A^T\mathbf{y} - \mathbf{s} &= -\mathbf{c}. \end{aligned}$$

$\min:\ 0 \cdot u$ $\quad Au = 0$ $\quad u \geq 0$

$\max\ 0^T y$ $\quad$ s.t. $A^T y + v = 0$ $\quad v \geq 0$

This system has total $2n + m$ unknowns and $2n + m$ equations including n nonlinear equations.

$u_j v_j = 0.$

Theorem 8 *If both (LP) and (LD) are feasible, there exists a strictly complementary solution pair $\mathbf{x}$ and $(\mathbf{y}, \mathbf{s})$ such that*

$$x_j + s_j > 0,\ \forall j.$$

Rules to Construct the Dual

obj. coef. vector	right-hand-side
right-hand-side	obj. coef. vector
A	A^T
Max model	Min model
$x_j \geq 0$ $\in K_j$	jth constraint $\geq$ $(K_j)^*$
$x_j \leq 0$	jth constraint $\leq$
x_j free	jth constraint $=$ 0
ith constraint $\leq$	$y_i \geq 0$
ith constraint $\geq$	$y_i \leq 0$
ith constraint $=$	y_i free

Conic LP

$$(CLP) \quad \text{minimize} \quad \mathbf{c} \bullet \mathbf{x}$$
$$\text{subject to} \quad \mathbf{a}_i \bullet \mathbf{x} = b_i, i = 1, 2, ..., m, \ \mathbf{x} \in C,$$

where C is a convex cone.

Linear Programming (LP): $\mathbf{c}, \mathbf{a}_i, \mathbf{x} \in \mathcal{R}^n$ and $C = \mathcal{R}^n_+$

Second-Order Cone Programming (SOCP): $\mathbf{c}, \mathbf{a}_i, \mathbf{x} \in \mathcal{R}^n$ and $C = SOC$

Semidefinite Programming (SDP): $\mathbf{c}, \mathbf{a}_i, \mathbf{x} \in \mathcal{S}^n$ and $C = \mathcal{S}^n_+$

Note that cone C can be a product of many (different) convex cones.

Dual of Conic LP

The dual problem to

$$(CLP) \quad \text{minimize} \quad \mathbf{c} \bullet \mathbf{x}$$
$$\text{subject to} \quad \mathbf{a}_i \bullet \mathbf{x} = b_i, i = 1, 2, ..., m, \ \mathbf{x} \in C.$$

is

$$(CLD) \quad \text{maximize} \quad \mathbf{b}^T \mathbf{y}$$
$$\text{subject to} \quad \sum_i^m y_i \mathbf{a}_i + \mathbf{s} = \mathbf{c}, \ \mathbf{s} \in C^*,$$

where $y \in \mathcal{R}^m$ are the dual variables, $\mathbf{s}$ is called the dual slack vector/matrix, and C^* is the dual cone of C.

Theorem 9 *(Weak duality theorem)*

$$\mathbf{c} \bullet \mathbf{x} - \mathbf{b}^T \mathbf{y} = \mathbf{x} \bullet \mathbf{s} \geq 0$$

for any feasible $\mathbf{x}$ *of (CLP) and* $(\mathbf{y}, \mathbf{s})$ *of (CLD).*

CLP Duality Theories

The weak duality theorem shows that a feasible solution to either problem yields a bound on the value of the other problem. We call $\mathbf{c} \bullet \mathbf{x} - \mathbf{b}^T\mathbf{y}$ the duality gap.

Corollary 1 *Let* $\mathbf{x}^* \in \mathcal{F}_p$ *and* $(\mathbf{y}^*, \mathbf{s}^*) \in \mathcal{F}_d$*. Then,* $\mathbf{c} \bullet x^* = \mathbf{b}^T\mathbf{y}^*$ *implies that* $\mathbf{x}^*$ *is optimal for (CLP) and* $(\mathbf{y}^*, \mathbf{s}^*)$ *is optimal for (CLD).*

Is the reverse also true? That is, given $\mathbf{x}^*$ optimal for (CLP), then there is $(\mathbf{y}^*, \mathbf{s}^*)$ feasible for (CLD) and $\mathbf{c} \bullet \mathbf{x}^* = \mathbf{b}^T\mathbf{y}^*$?

This is called the Strong Duality Theorem and it is “true” for LP, but it is “False” in general cases.

When Strong Duality Theorems Holds for CLP

Theorem 10 *(Strong duality theorem) Let $\mathcal{F}_p$ and $\mathcal{F}_d$ be non-empty and at least one of them has an interior. Then, $\mathbf{x}^*$ is optimal for (CLP) and $(\mathbf{y}^*, \mathbf{s}^*)$ is optimal for (CLD) if any only if*

$$\mathbf{c} \bullet \mathbf{x}^* = \mathbf{b}^T \mathbf{y}^*.$$

x^* attainable

There are cases that the duality gap tends to zero but the optimal solution is not attainable.

More Duality Theorem for CLP

Theorem 11 *(CLP duality theorem) If one of (CLP) or (CLD) is unbounded then the other has no feasible solution.*

If (CLP) and (CLD) are both feasible, then both have bounded optimal objective values and the optimal objective values may have a duality gap.

If one of (CLP) or (CLD) has a strictly or interior feasible solution and it has an optimal solution, then the other is feasible and has an optimal solution with the same optimal value.

Optimality Conditions for SDP

$$\begin{array}{rcl} \mathbf{c} \bullet X - \mathbf{b}^T \mathbf{y} & = & 0 \\ \mathcal{A} X & = & \mathbf{b} \\ -\mathcal{A}^T \mathbf{y} - S & = & -\mathbf{c} \\ X, S & \succeq & \mathbf{0} \end{array}, \tag{1}$$

$$\begin{array}{rcl} XS & = & \mathbf{0} \\ \mathcal{A} X & = & \mathbf{b} \\ -\mathcal{A}^T \mathbf{y} - S & = & -\mathbf{c} \\ X, S & \succeq & \mathbf{0} \end{array} \tag{2}$$

Rank of SDP Solutions

At any optimal solution pair (X^*, S^*)

$$\text{rank}(X^*) + \text{rank}(S^*) \leq n.$$

If the equality holds, they are a strictly complementary solution pair.

There are optimal solutions of X^* and S^* such that the rank of X^* and the rank of S^* are minimal, respectively.

There are optimal solutions of X^* and S^* such that the rank of X^* and the rank of S^* are maximal, respectively.

Rank Reduction Methods:...

uniqueness

SNL: System of Quadratic Equations

System of nonlinear equations for $\mathbf{x}_i \in R^d$:

$$\|\mathbf{x}_i - \mathbf{x}_j\| = d_{ij}, \; \forall\,(i,j) \in N_x, \; i < j,$$
$$\|\mathbf{a}_k - \mathbf{x}_j\| = d_{kj}, \; \forall\,(k,j) \in N_a,$$

where $\mathbf{a}_k$ are possible points whose locations are known, often called anchors.

One can equivalently represent it as

$$\|\mathbf{x}_i - \mathbf{x}_j\|^2 = d_{ij}^2, \; \forall\,(i,j) \in N_x, \; i < j,$$
$$\|\mathbf{a}_k - \mathbf{x}_j\|^2 = d_{kj}^2, \; \forall\,(k,j) \in N_a,$$

which becomes a system of multi-variable-quadratic equations.

Matrix Representation of SNL

Let $X = [\mathbf{x}_1\ \mathbf{x}_2\ \dots\ \mathbf{x}_n]$ be the $2 \times n$ matrix that needs to be determined. Then

$$\|\mathbf{x}_i - \mathbf{x}_j\|^2 = (\mathbf{e}_i - \mathbf{e}_j)^T X^T X (\mathbf{e}_i - \mathbf{e}_j) \text{ and}$$
$$\|\mathbf{a}_k - \mathbf{x}_j\|^2 = (\mathbf{a}_k; -\mathbf{e}_j)^T [I\ \ X]^T [I\ \ X] (\mathbf{a}_k; -\mathbf{e}_j),$$

where $\mathbf{e}_i$ is the vector with 1 at the ith position and zero everywhere else.

$$\begin{aligned}
(\mathbf{e}_i - \mathbf{e}_j)^T (\mathbf{e}_i - \mathbf{e}_j) \bullet Y &= d_{ij}^2,\ \forall\, i, j \in N_x,\ i < j, \\
(\mathbf{a}_k; -\mathbf{e}_j)^T (\mathbf{a}_k; -\mathbf{e}_j) \bullet \begin{pmatrix} I & X \\ X^T & Y \end{pmatrix} &= d_{kj}^2,\ \forall\, k, j \in N_a, \\
Y - X^T X &= \mathbf{0}.
\end{aligned}$$

where Y denotes the Gram matrix $X^T X$.

SDP Relaxation

Change

$$Y - X^T X = \mathbf{0}$$

n×d \ d×n

to

d=2

$$Y - X^T X \succeq \mathbf{0}.$$

This matrix inequality is equivalent to

$\Updownarrow$ ✓

(d+n)×(d+n)

$$Z := \begin{pmatrix} I & X \\ X^T & Y \end{pmatrix} \succeq \mathbf{0}.$$

This is the semidefinite matrix cone, and the problem becomes an SDP Feasibility problem.

SDP Feasibility Standard Form

Find a symmetric matrix $Z \in \mathbf{R}^{(2+n)\times(2+n)}$ such that

$$\begin{aligned}
Z_{1:2,1:2} &= I \\
(\mathbf{0}; \mathbf{e}_i - \mathbf{e}_j)(\mathbf{0}; \mathbf{e}_i - \mathbf{e}_j)^T \bullet Z &= d_{ij}^2, \ \forall\, i, j \in N_x,\ i < j, \\
(\mathbf{a}_k; -\mathbf{e}_j)(\mathbf{a}_k; -\mathbf{e}_j)^T \bullet Z &= d_{kj}^2, \ \forall\, k, j \in N_a, \\
Z &\succeq \mathbf{0}.
\end{aligned}$$

This is semidefinite programming feasibility system (with a null objective).

When this relaxation is exact?

One case is that the single unknown point $\mathbf{x}_1$ is connected to three anchors $\mathbf{a}_k$, $k = 1, 2, 3$.

In general, if the rank of a feasible Z is 2, then it solves the original graph relaxation problem.

Duality Theorem for SNL

Theorem 12 *Let $\bar{Z}$ be a feasible solution for SDP and $\bar{U}$ be an optimal slack matrix of the dual. Then,*

1. *complementarity condition holds: $\bar{Z} \bullet \bar{U} = 0$ or $\bar{Z}\bar{U} = \mathbf{0}$;*
2. *$(\bar{Z}) + (\bar{U}) \leq 2 + n$;*
3. *$(\bar{Z}) \geq 2$ and $(\bar{U}) \leq n$.*

An immediate result from the theorem is the following:

Corollary 2 *If an optimal dual slack matrix has rank n, then every solution of the SDP has rank 2, that is, the SDP relaxation solves the original problem exactly.*

Basic Feasible Solution of LP

In the LP standard form when A has a full row-rank, select m linearly independent columns, denoted by the index set B, from A. Solving the m-dimension vector $\mathbf{x}_B$ from

$$A_B \mathbf{x}_B = \mathbf{b}$$

and setting the rest variables, denoted by $\mathbf{x}_N$, to zero, we obtain a solution $\mathbf{x}$ such that

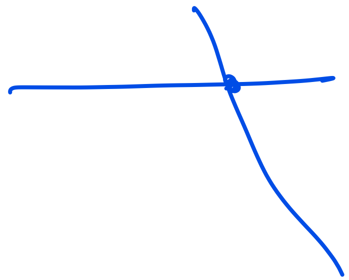
$$A\mathbf{x} = \mathbf{b}.$$

Then, $\mathbf{x}$ is said to be a (primal) basic solution to (LP) with respect to the basis A_B. The entries in $\mathbf{x}_B$ are called basic variables. If a basic solution $\mathbf{x} \geq \mathbf{0}$, then $\mathbf{x}$ is called a basic feasible solution.

If one or more components in $\mathbf{x}_B$ have zero value, then the basic feasible solution $\mathbf{x}$ is said to be (primal) degenerate.

The basic feasible solution is an extreme point of the feasible region. In general (i.e., A has no full row-rank), any solution from $A_B \mathbf{x}_B = \mathbf{b},\ \mathbf{x}_B \geq \mathbf{0}$, where columns of A_B are linearly independent, is an extreme point of the feasible region.

A dual vector $\mathbf{y}$ satisfying



$$A_B^T \mathbf{y} = \mathbf{c}_B$$

is said to be the corresponding dual basic solution.

If the dual basic solution is also feasible, that is,

$$\mathbf{s} = \mathbf{c} - A^T \mathbf{y} \geq \mathbf{0}$$

, then it is a dual Basic Feasible Solution.

If one or more slacks in $\mathbf{c}_N - A_N^T \mathbf{y}$ have zero value, that dual basic feasible solution $\mathbf{y}$ is said to be (dual) degenerate.

The dual basic feasible solution is an extreme point of the dual feasible region. In general (i.e., A has no full row-rank), any solution from $A_B^T \mathbf{y} = \mathbf{c}_B, \; \mathbf{c} - A^T \mathbf{y} \geq \mathbf{0}$, where rows of A_B are linearly independent, is an extreme point of the dual feasible region.

Theorem 13 *(LP fundamental theorem) Given (LP) and (LD) where A has full row rank m,*

i) *if there is a feasible solution, there is a basic feasible or extreme solution ;*

ii) *if there is an optimal solution, there is an optimal basic feasible or extreme solution.*

Sample Problem 1

Let $A_1 \in R^{m \times n}$, $A_2 \in R^{m \times p}$ be two given matrices, and let $\mathbf{c}_1 \in R^n$, $\mathbf{c}_2 \in R^p$ be two given *non-negative vectors*. Consider the problem

$$z^* \left\{ \begin{array}{ll} \min & \mathbf{c}_1^T \mathbf{x}_1 + \mathbf{c}_2^T \mathbf{x}_2 \\ \text{s.t.} & A_1 \mathbf{x}_1 + A_2 \mathbf{x}_2 = \mathbf{b} \\ & \mathbf{x}_1, \quad \mathbf{x}_2 \geq \mathbf{0}, \end{array} \right.$$

$A_1^T y \le c_1$

$A_2^T y \le c_2$

and assume the problem is *feasible*.

(a) Sow that the problem has an optimal solution $(\mathbf{x}_1^*, \mathbf{x}_2^*)$.

(b) Let $(\mathbf{x}_1, \mathbf{x}_2)$ be a feasible solution to the problem and its objective value equals $\mathbf{b}^T y$ where $\mathbf{y}$ satisfies

$$A_1^T \mathbf{y} \leq \alpha_1 \mathbf{c}_1 \qquad \alpha_1 \geq \phi$$

$$A_2^T \mathbf{y} \leq \alpha_2 \mathbf{c}_2, \qquad \alpha_2 \geq 1$$

where α_1 and α_2 are two scalars greater than or equal to 1, then

$$\mathbf{c}_1^T\mathbf{x}_1 + \mathbf{c}_2^T\mathbf{x}_2 = \mathbf{b}^T\mathbf{y} \leq \overset{1.1}{\alpha_1} \cdot \mathbf{c}_1^T\mathbf{x}_1^* + \overset{1.2}{\alpha_2} \cdot \mathbf{c}_2^T\mathbf{x}_2^*. \quad \leq \boxed{\max\{\alpha_1, \alpha_2\} \cdot} (c_1^T x_1^* + c_2^T x_2^*)$$

(α_1, α_2) is usually called the bi-factor approximation ratio and used in approximating algorithms for $= z^*$
combinatorial optimization.

$\leq 1.2 \cdot z^*$

(a) Consider the dual problem:

$$\begin{array}{ll} \max & \mathbf{b}^T\mathbf{y} \\ \text{subject to} & A_1^T\mathbf{y} \le \mathbf{c}_1, \\ & A_2^T\mathbf{y} \le \mathbf{c}_2. \end{array}$$

Since $\mathbf{c}_1 \ge \mathbf{0}$ and $\mathbf{c}_2 \ge \mathbf{0}$, $\mathbf{y} = \mathbf{0}$ is a feasible point for the dual. By LP duality, since both the primal and dual problems are feasible, both must have optimal solutions.

(b) Let $(\mathbf{x}_1^*, \mathbf{x}_2^*)$ be a primal optimal solution, and let $(\mathbf{x}_1, \mathbf{x}_2)$ be a primal feasible solution with value $c\_1^T\mathbf{x}_1 + \mathbf{c}_2^T\mathbf{x}_2 = \mathbf{b}^T\mathbf{y}$, where $\mathbf{y}$ satisfies

$$A_1^T\mathbf{y} \leq \alpha_1\mathbf{c}_1, \quad A_2^T\mathbf{y} \leq \alpha_2\mathbf{c}_2.$$

Since $(\mathbf{x}_1^*, \mathbf{x}_2^*)$ is primal feasible, $A_1\mathbf{x}_1^* + A_2\mathbf{x}_2^* = \mathbf{b}$, and $(\mathbf{x}_1^*, \mathbf{x}_2^*) \geq \mathbf{0}$. Then,

$$(A_1^T\mathbf{y})^T\mathbf{x}_1^* \leq \alpha_1\mathbf{c}_1^T\mathbf{x}_1^*,$$

$$(A_2^T\mathbf{y})^T\mathbf{x}_2^* \leq \alpha_2\mathbf{c}_2^T\mathbf{x}_2^*.$$

Sum up all these inequalities, we have

$$\mathbf{b}^T\mathbf{y} = (A_1\mathbf{x}_1^* + A_2\mathbf{x}_2^*)^T\mathbf{y} = (A_1^T\mathbf{y})^T\mathbf{x}_1^* + (A_2^T\mathbf{y})^T\mathbf{x}_2^* \leq \alpha_1\mathbf{c}_1^T\mathbf{x}_1^* + \alpha_2\mathbf{c}_2^T\mathbf{x}_2^*.$$

Sample Problem 2

Consider the standard LP or SDP primal and dual pair. Prove that, if both of them have interior feasible solutions, the optimal solution set is bounded.

Initial $A\overset{0}{x}=b,\ \overset{0}{x}>0,\quad A^T\overset{0}{y}+\overset{0}{s}=c,\ \overset{0}{s}>0$

Optim $Ax^*=b,\ x^*\ge 0 \quad - \quad A^Ty^*+s^*=c,\ s^*\ge 0$

$s^0-s^*=-A^T(y^0-y^*)$

$0=(x^0-x^*)^T(s^0-s^*)= (x^0)^Ts^0-(s^0)^Tx^*-(x^0)^Ts^*+(s^*)^Tx^*$

$=(x^0-x^*)^TA^T(y^*-y^0)$

$=(A(x^0-x^*))^T(y^*-y^0)$

$=(b-b)^T(y^*-y^0)$

$=(x^0)^Ts^0-(s^0)^Tx^*-(x^0)^Ts^*$

$\Rightarrow (s^0)^Tx^*+(x^0)^Ts^*=\boxed{(x^0)^Ts^0}$

Sample Problem 3

TA. Saturday
Ye. Sunday
1-3 pm

Consider the RL/MDP fixed-point computation by linear program:

$$
\begin{array}{llll}
\text{maximize}_{\mathbf{y}} & \sum_{i=1}^m y_i & & \\
\text{subject to} & y_1 - \gamma \mathbf{p}_j^T \mathbf{y} & \le & c_j,\ j \in \mathcal{A}_1 \\
& & \vdots & \\
& y_i - \gamma \mathbf{p}_j^T \mathbf{y} & \le & c_j,\ j \in \mathcal{A}_i \\
& & \vdots & \\
& y_m - \gamma \mathbf{p}_j^T \mathbf{y} & \le & c_j,\ j \in \mathcal{A}_m .
\end{array}
$$

$e^T \{ (e-\gamma e)^T x_1 + (e-\gamma e)^T x_2 \} = (1-\gamma)(e^T x_1 + e^T x_2)$

$(I-\gamma P_1)x_1 + (I-\gamma P_2)x_2 = e$, $e^T(\ldots) = m$

$x_1 \ge 0,\ x_2 \ge 0$

$e^T x_1 + e^T x_2 = \frac{m}{1-\gamma}$

Red, Blue

$c_1^T x_1 + c_2^T x_2$

$(I-\gamma P_1)x_1 + (I-\gamma P_2) = e$

$x_1 \ge 0,\ x_2 \ge 0$

$e^T P_1 = e,\ e^T P_2 = e$

Prove at the optimal solution $\mathbf{y}^*$:

$$y_i^* = \min_{j \in \mathcal{A}_i} \{ c_j + \gamma \mathbf{p}_j^T \mathbf{y}^* \} \ \forall i$$

What is the dual of the problem? What is the sum of a feasible solution of the dual?