MS&E 310 Course Project: Prediction Market

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The prediction market (also called information market) is a fast developing area of research. Empirical studies have shown that the wisdom of crowds have a stronger prediction power than the other prediction alternatives, such as taking polls or asking experts' opinions. In this course project, you are asked to study and explore some theories of the prediction market as well as implement it. Your data will be from the course website, which is the game you played throughout this quarter.

In the following, please try your best to address all the questions. For most of the questions, you can find useful information on the papers we referred. Also, you are welcomed to use any references you like, but make sure to cite them when you use.

The background of this project is the football prediction market game you played throughout this quarter. Assume that there are m states of futures of which exactly one will happen. Each bidder is allowed to bid on any subset of the states, with a price limit and a quantity limit. In what follows, we use $\mathbf{a}_j$ to denote the bid profile of the jth bidder, π_j to denote his price limit and q_j to denote his share quantity limit. The call auction model we studied in class (see course Slide #2 and the supplement slides on information market where indices i and j are switched) is as follows:

$$\begin{array}{lll} \text{maximize}_{\mathbf{x},y} & \left(\sum_j \pi_j x_j\right) - y & \\ \text{s.t.} & \sum_j a_{ij} x_j \le y, \ \forall\ i = 1, 2, ..., m & \qquad (1) \\ & 0 \le x_j \le q_j, \ \forall\ j = 1, ..., n. & \end{array}$$

Question 1: Study the above call auction model and show the dual price of each state. Download the data for the first and second game from the website and solve these two markets according to the call auction model (use CVX, Matlab LP, or the Excel solver). Submit the dual price of the first two games and solver files.

Question 2: Give an example with less than 3 bids and 3 states, showing that the state price vector may not be unique.

Question 3: In order to get a unique price, people sometimes add another term to the objective function when solving the call auction model; one model which is called CPCAM is defined as follows:

$$\begin{array}{ll} \text{maximize}_{\mathbf{x},y,\mathbf{s}} & \sum_j \pi_j x_j - y + u(\mathbf{s}) \\ \text{s.t.} & \sum_j a_{ij} x_j + s_i = y, \ \forall\ i = 1,2,...,m, \\ & 0 \le x_j \le q_j, \ \forall\ j = 1,...,n, \\ & s_i \ge 0, \forall\ i = 1,...,m. \end{array} \tag{2}$$

where $u(\mathbf{s}) = u(s_1, \ldots, s_m)$ is increasing, and strictly concave. Write down its dual problem and the KKT conditions, argue why this problem has unique price. How would you interpret this $u(\mathbf{s})$? (Hint: Read [1] and [3])

Question 4: The disadvantage of the call auction model is that it can't tell the bidders whether their bids are accepted or not until the market closes. This is undesirable since sometimes the bidders want to know the results to their bids immediately so that they can modify their bids and submit again. Therefore, in practice, the market is usually implemented in an online version which is defined as follows. Instead of solving the optimization after the market closes, whenever kth bidder submits a bid, the market maker solves the following optimization problem:

$$\begin{array}{ll} \text{maximize}_{x_k,y,\mathbf{s}} & \pi_k x_k - y + u(\mathbf{s}) \\ \text{s.t.} & \sum_{j=1}^{k} a_{ij} x_j + s_i = y, \ \forall\ i = 1,2,...,m \\ & 0 \le x_k \le q_k \\ & s_i \ge 0, \ \forall\ i = 1,...,m. \end{array} \tag{3}$$

Note that, in (3), only x_k, y and $\mathbf{s}$ are variables, and quantity $x_j, j < k$ is already "locked down". Here again we assume that u is increasing and strictly concave. Write down the dual problem of (3) and show that it can be solved efficiently.

Question 5: We continue our discussion in Question 4. Assume that the kth bidder has valuation $\hat{\pi}$ for his bid. We solve (3) for the optimal x_k^*. Then we charge the bidder by

$$\chi(0) - \chi(x_k^*)$$

where

$$\chi(b) = \begin{array}{ll} \text{maximize}_{y,\mathbf{s}} & -y + u(\mathbf{s}) \\ \text{s.t.} & s_i = y - q_i - a_{ik} b, \ \forall\ i = 1,2,...,m. \end{array} \tag{4}$$

where $q_i = \sum_{j=1}^{k-1} a_{ij} x_j$ is the outstanding shares of state i that has been already accepted.

Prove under this charging method, the optimal strategy for this bidder is to bid $\hat{\pi}$ (Assume $\mathbf{a}_k$ and q_k is fixed).

A mechanism is truthful (or incentive compatible) if and only if for every agent, reporting his truthful belief is always the optimal strategy, disregarding what others do (that is also called dominant strategy in game theory). Therefore, the above strategy is truthful. In fact, the above mechanism is a special case of the VCG mechanism. Read references [3] and [4]. Show a proof of truthfulness using the VCG framework.

Question 6: Run the market with model (3) with utility functions:

$$u_1(\mathbf{s}) = \frac{b}{m}\sum_i \log s_i$$

and

$$u_2(\mathbf{s}) = \frac{b}{m}\sum_i (1 - e^{-s_i})$$

separately for a parameter $0 < b < 1$.

The bids come in the timely order. Compare the results generated from the two on-line models and their off-line call auction models using simulation data. Which one accepts more bids and which one has better prediction power? Interpret your observations and findings.

Question 7: Consider the off-line LP problem

$$\begin{array}{lll} (LP) & \text{maximize} & \pi^T \mathbf{x} = \sum_{j=1}^n \pi_j x_j \\ & \text{subject to} & \sum_{j=1}^n \mathbf{a}_j x_j = A\mathbf{x} \leq \mathbf{b},\ \mathbf{0} \leq \mathbf{x} \leq \mathbf{e}; \end{array}$$

where data $A \in R^{m\times n}$, $\mathbf{a}_j \in R^m$ (each entry is either 0 or 1), $\mathbf{c} \in R^n$, $\mathbf{b} \in R^m$ and $\mathbf{e}$ is the vector of all ones, and variables $\mathbf{x} \in R^n$. You may interpret this is a linear program to allocated the items of inventory $\mathbf{b}$ (assuming large quantities) to n customers such that the revenue is maximized.

Now the variables come in the timely order. Design an on-line mechanism to sell the items such that the revenue is maximized, and compare it to the off-line model either by theory or simulation.

All other observations, findings and comments are welcome.

References

[1] M. Peters. CONVEX MECHANISMS FOR PARI-MUTUEL MARKETS. Ph.D. Thesis, Stanford, November 2008.

[2] M. Peters, A. M-C. So and Y. Ye. Pari-mutuel Markets: Mechanisms and Performance. *The 3rd International Workshop On Internet And Network Economics*, 2007.

[3] S. Agrawal, E. Delage, M. Peters, Z. Wang and Y. Ye. A Unified Framework for Dynamic Pari-mutuel Information Market Design. *The 10th ACM Conference on Electronic Commerce*, 2009.

[4] $http://en.wikipedia.org/wiki/Vickrey\_auction.$

[5] N. Nisan, T. Roughgarden, E. Tardos and V. Vazirani. Algorithmic Game Theory Cambridge University Press, New York, NY, USA.