MS&E 310 **Midterm Examination**
Linear Programming **October, 2012**

DUE: in 24 hours

Note: No late exams will be accepted. In taking this examination, you acknowledge and accept the Stanford University Honor Code.

Question 1 (30pts). **Portfolio Management for World Cup Assets.**

You had this problem in HW1. Suppose there are 5 assets/securities available in the World Cup Market for open trading at fixed prices and pay-offs; see the table below. Here, for example, Security 1's pay-off is \$1 if either Argentina, Brazil, or Italy wins. The Share Limit represent the maximum number of shares one can purchase, and Price is the current purchasing price per share of each security.

Security	Price π	Share Limit q	Argentina	Brazil	Italy	Germany	France
1	\$0.75	10	\$1	\$1	\$1		
2	\$0.35	5				\$1	
3	\$0.40	10	\$1		\$1		\$1
4	\$0.95	10	\$1	\$1	\$1	\$1	
5	\$0.75	5		\$1		\$1	

(a) Assume that short is not allowed, that is, one can only buy shares not sell. Formulate the problem to decide how many shares of each security to purchase so as to maximize the worst-case (minimum) pay-off when the game is finally realized.

(b) Form the dual of (a) and give an interpretation of the dual. What does the duality and/or complementarity theorem imply for the optimal solution of the problem?

(c) Now assume there is no share limit (can be ∞) and short is allowed, that is, the decision variable can be both positive (buy) and negative (sell). Reformulate the portfolio problem to decide how many shares of each security to purchase so as to maximize the worst-case (minimum) pay-off when the game is finally realized.

(d) When there is no share limit and short is allowed, an arbitrage opportunity is a feasible solution to the problem in part (c) that gives a positive objective value. A risk neutral probability p is a vector satisfying $e^T p = 1$ such that the price of each asset equals its expected payoff under p. Prove the absence of arbitrage is equivalent to the existence of a risk neutral probability. This is actually the first fundamental theorem in asset pricing theory.

Question 2 (20pts). Use any LP solver to solve the following linear program and find the optimal basis:

$$
\begin{aligned}
\min \quad & 2x_1 - 3x_2 + x_3 \\
\text{s.t.} \quad & x_1 + 4x_2 - 2x_3 = 4 \\
& -x_1 + 5x_2 - x_3 \leq 6 \qquad (1) \\
& x_1, x_2, x_3 \geq 0 \qquad (2)
\end{aligned}
$$

(a) What is the range for the right-hand-side value 4 such that the optimal basis remains optimal?

(b) The range for the objective coefficient x_1 such that the optimal basis remains optimal?

(c) The range for the objective coefficient x_2 such that the optimal basis remains optimal?

Question 3 (30 pts). Consider the LP game where a finite set K of firms each of whom has operations that have representations as linear programs. Suppose the linear program representing the operations of each firm k in K entails choosing an n-dimension vector $x \geq 0$ of activity levels that maximize the firm's profit

$$c^T x$$

subject to the constraint that its consumption $A^{\{k\}}x$ of resources minorizes its resource vector $b^{\{k\}} \in R^m$, that is,

$$A^{\{k\}}x \leq b^{\{k\}}.$$

Here $A^{\{k\}} \geq 0$ for all $k \in K$, and it is the so called resource/product consumption matrix for firm k. For example, $a_{ij}^{\{k\}}$ represents the consumption rate of the ith resource per unit of the jth product for firm k. Different firms may have different $a_{ij}^{\{k\}}$ since they may have different technologies to produce the same product j using the same resource i. The smaller $a_{ij}^{\{k\}}$, the more advanced the technology.

An alliance is a subset of the firms. If an alliance S pools its resource vectors, the linear program that S faces is that of choosing an n-dimension vector $x \geq 0$ that maximizes the profit $c^T x$ that S earns subject to its resource constraint

$$A^S x \leq b^S,$$

where

$$A^S = (a_{ij}^S) = (\min_{k \in S}\{a_{ij}^{\{k\}}\}) \quad \text{and} \quad b^S = \sum_{k \in S} b^{\{k\}},$$

that is, the alliance will adopt the most efficient technologies among its members for each (i, j) and utilize all their resources. For example, if

$$A^{\{1\}} = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \quad \text{and} \quad A^{\{2\}} = \begin{pmatrix} 2 & 1 \\ 3 & 3 \end{pmatrix}$$

and

$$b^{\{1\}} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad \text{and} \quad b^{\{2\}} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

Then

$$A^{\{1,2\}} = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} \quad \text{and} \quad b^{\{1,2\}} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}.$$

Let V^S be the resulting maximum profit of alliance S:

$$\begin{array}{rll} V^S := & \max & c^T x \\ & \text{s.t.} & A^S x \le b^S, \\ & & x \ge 0. \end{array}$$

The grand alliance is the set $S = K$ of all firms.

Core is the set of payment vector $z = (z_1, \ ..., \ z_{|K|})$ to each firm such that

$$\sum_{k \in K} z_k = V^K$$

and

$$\sum_{k \in S} z_k \ge V^S, \ \forall S \subset K.$$

1. Show that the core is a convex set.
2. Write out the dual of the grand alliance problem.
3. Prove that for each optimal dual price vector for the linear program of the grand alliance, allocating each firm the value of its resource vector at those prices yields a profit allocation in the core.
4. Construct a simple example where a core payment is not necessarily drawn from the dual price vector of the grand alliance problem.

Question 4 (20pts). Assume that all basic feasible solutions (BFS) of a standard LP problem are non degenerate (that is, every basic variable has a positive value at every BFS). Then consider using the Simplex method to solve the problem. Prove that, if at a pivot step there is exactly one negative reduced cost coefficient, then the corresponding entering variable will remain as a basic variable for the rest steps of the Simplex method.