MS&E 239: Computational Advertising
Homework 2

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1. A honey bee visits flowers placed on the number line, at each integer points $0, \pm 1, \pm 2, \ldots$. It starts at the flower in position 0 and in each time period, it either moves one step left or right, each with probability 0.5. What is the probability that the bee is at the flower in position k after $n = 10$ steps? What about $n = 20$ steps?

   A. Fix $n = 10$. If $|k| > 10$, then there is no way for the bee to reach the flower in position k starting from 0 in 10 steps, and hence the corresponding probability is 0.

   As the total number of steps taken by the bee is even, the number of steps taken to the left is same as the number taken to the right. Hence, after $n = 10$ steps, the bee must be at a flower in an even position. Thus, for odd k, again the probability is zero.

   Finally, suppose k is even, with $|k| \leq 10$. Suppose the bee has taken n_1 steps to the right and n_2 steps to the left. As the total number of steps taken is n, we have

$$n_1 + n_2 = n.$$

   For the bee to be at the flower in position k, it must be that the excess number os steps to the right is equal to k, i.e., we have

$$n_1 - n_2 = k.$$

   Together, this implies that $n_1 = (n + k)/2$ and $n_2 = (n - k)/2$. Thus, for the bee to be in position k, it has to take n_1 steps to the right, and n_2 steps to the left. The total number of paths that achieve this is $\binom{n}{n_1}$, and the probability of any path is $(0.5)^n$. Hence, for even k with $|k| \leq 10$, the probability the bee is in the flower at position k is given by

$$\binom{n}{n_1} \left(\frac{1}{2}\right)^n = \binom{n}{\frac{n+k}{2}} \left(\frac{1}{2}\right)^n.$$

   A similar argument follows for $n = 20$.

2. A publisher has a total inventory of 200 impresssions, that she has allocate between a GD contract and the NGD market. The GD contract has a demand of 50 impressions. In the NGD market, the price density (i.e. density of the highest bid) is given by

$$f(p) = 2(1 - p), \text{for } p \in [0, 1].$$

   How should the publisher allocate the impressions between the NGD market and the GD contract under the following conditions?

   (a) The publisher seeks to maximize her revenue.

A. As the publisher wants to maximize her revenue, she should send only the low priced ads to the GD market. This can be achieved by allocating all ads with price above a certain threshold to the NGD market. The threshold is chosen so that the supply to the GD contract is 50.

Suppose the threshold is c. Then, the supply S_{GD} to the GD market is given by

$$\begin{aligned} S_{GD} &= 200 \int_0^c f(p)dp \\ &= 200(1 - (1-c)^2). \end{aligned}$$

As we need $S_{GD} = 50$, we have $1 - (1-c)^2 = 50/200 = 1/4$. This implies that $(1-c)^2 = 3/4$ and hence

$$\begin{aligned} c &= 1 - \frac{\sqrt{3}}{2} \\ &\simeq 0.13397. \end{aligned}$$

Hence, the publisher should send only those ads with prices below $c \simeq 0.13397$ to the GD contract. This can be achieved by setting a reserve of $c \simeq 0.13397$ in the NGD market.

(b) The publisher wants the allocate the "best" ads to the GD contract. (Assume higher priced ads are better).

A. Here the publisher wants to allocate the higher priced ad to the GD market, and hence chooses a threshold price $x < 1$ such that she allocates all ads priced above x to the GD contract. The value of x is chosen so that the total expected supply to the GD contract is equal to 50.

In this case, the supply to the GD contract is given by

$$\begin{aligned} S_{GD} &= 200 \int_x^1 f(p)dp \\ &= 200(1-x)^2. \end{aligned}$$

As $S_{GD} = 50$, we have $(1-x)^2 = 50/200$, and hence $x = 0.5$.

This allocation rule is harder to implement, as the publisher can only see the price of an ad after she has sent the ad to the NGD marketplace. One possible way is to set a very high reserve of 1, and then voluntarily assign the ad to the second highest bidder if her bid is lower than x.

(c) The publisher wants to allocate the inventory so that the price distribution in GD mirrors the intrinsic distribution in the total supply.

A. For the price distribution in GD to mirror the intrinsic distribution in the total supply, the publisher has to allocate $50/200 = 1/4$ of the ads in each price level to the GD contract and the remaining to the NGD contract. One allocation rule that achieves this is to randomly and independently assign each ad with probability $1/4$ to the GD contract and with remaining probability to the NGD contract.