MS&E 211 Fall 2009
Linear and Nonlinear Optimization Oct. 15, 2009
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Homework Assignment 3: Due 3:15pm Monday, Oct. 26

Problem 1. Consider the Parimutuel Digital Call Auction (PDCA), Problem 6 of Homework 2. In Part 1, we modeled the problem as LP to decide the contract award vector $\mathbf{x}$ and to price each state using the expected payback. The model that we had was:

$$\begin{aligned} \text{Maximize} \quad & \pi^T\mathbf{x} - \mathbf{v}^T\mathbf{s} \\ \text{subject to:} \quad & A\mathbf{x} - \mathbf{s} = 0 \\ & \mathbf{x} \le \mathbf{q} \\ & \mathbf{x} \ge \mathbf{0} \end{aligned}$$

Set price vector $\mathbf{p}$ as dual values for constraint $A\mathbf{x} - \mathbf{s} = \mathbf{0}$. Explain the price properties using duality and/or complementarity.

Problem 2. Consider the following LP formulation:

$$\begin{aligned} \text{maximize} \quad & 2x_1 + 4x_2 + x_3 + x_4 \\ \text{subject to:} \quad & x_1 + 3x_2 + x_4 \le 4 \\ & 2x_1 + x_2 \le 3 \\ & x_2 + 4x_3 + x_4 \le 3 \\ & x_1, x_2, x_3, x_4 \ge 0 \end{aligned}$$

Answer the following questions with the help of the final simplex tableau.

(a) Will the optimal basis change if we respectively change b =(4, 3, 3) to (5, 4, 4), (3, 2, 2) or (2, 1, 1)?
(b) Will the optimal basis change if we respectively change c =(2, 4, 1, 1) to (3, 4, 1, 1), (2, 3, 1, 1), (2, 4, 10, 1) or (2, 4, 1, 2)?
(c) What happens to the optimal value for a small change in b, say b becomes (5, 2, 3)? How can you quickly calculate the change in the optimal objective function value? what is the new objective value?

(d) What happens to the optimal value for a small change in c, say c becomes (2.5, 4, 2, 1)? How can you quickly calculate the change in the optimal objective function value? what is the new objective value?

Problem 3. Multi-Firm Alliance Problem

There are 3 firms A, B, and C in this problem. The common profit margin vector c is (1; 2; 4). The resource computation rate matrix is given by:

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

There are three available resources during production. The resource vector for company A, B and C are given separately by (1; 2;3), (2; 3; 1) and (3; 2; 1).

(a) What is the profit for the grand alliance? Explain why the alliance is preferred in this problem.

(b) What is the core of the problem? Solve it by checking the dual problem. Verify that the profit allocation derived from the core is desirable.

Problem 4. There is in general a strong connection between the theories of optimization and free market competition, which is illustrated by an idealized model of activity location. Suppose there are 3 economic activities (various factories, homes, store, etc.) that are to be individually located on 3 distinct parcels of land. If activity i is located on parcel j that activity can yield s_{ij} units (dollars) of value.

If the assignment of activities to land parcels is made by a central authority, it might be made in such a way as to maximize the total value generated. In other words, the assignment would be made so as to maximize $\sum_i \sum_j s_{ij} x_{ij}$ where:

$$x_{ij} = \begin{cases} 1 & \text{if activity } i \text{ is assigned to parcel } j \\ 0 & \text{otherwise} \end{cases}$$

More explicitely, this approach leads to the optimization problem:

$$\begin{aligned} \text{maximize } & \sum_i \sum_j s_{ij} x_{ij} \\ \text{subject to: } & \sum_j x_{ij} = 1 \quad i = 1, 2, 3 \\ & \sum_i x_{ij} = 1 \quad j = 1, 2, 3 \\ & x_{ij} \geq 0, \quad x_{ij} = 0 \text{ or } 1 \end{aligned}$$

Actually, it can be shown that the final requirement ($x_{ij} = 0$ or 1) is automatically satisfied at any extreme point of the set defined by the other constraints.
If one considers the problem from the viewpoint of free competition, it is assumed that, rather than a central authority determining the assignment, the individual activities bid for the land and thereby establish prices.

(a) Show that there exists a set of activity prices p_i, $i = 1, 2, 3$ and land prices q_j, $j = 1, 2, 3$ such that

$$p_i + q_j \geq s_{ij} \quad i = 1, 2, 3, \; j = 1, 2, 3$$

with equality holding if in an optimal assignment activity i is assigned to parcel j.

(b) Show that Part (a) implies that if activity i is optimally assigned to parcel j and if j' is any other parcel

$$s_{ij} - q_j \geq s_{ij'} - q_{j'}$$

Give an economic interpretation of this result and explain the relation between free competition and optimality in this context.

(c) Assuming that each s_{ij} is positive, show that the prices can all be assumed to be non-negative.

Problem 5. Asset Allocation Duality

Suppose you are given \$10M to invest on start-up companies such that you maximize the total return. There are a fixed number of companies in which you can choose to invest your assets. Each company requires some fixed amount of capital and has a promised return. For each company, you must decide how much of its required capital to provide - somewhere from 0 to 100% (you can assume that other investors will provide the rest of the required capital). Then your return for that part of your total investment will be the companys promised return times the percent of its capital that you provided.

We model this problem as a LP. The primal and dual problems are:

$$\begin{array}{rll} & \textbf{Primal} & \textbf{Dual} \\ \text{maximize} & \sum_i p_i x_i & \text{minimize } My + sum_i \lambda_i \\ \text{subject to:} & \sum_i r_i x_i \leq M & r_i y + \lambda_i \geq p_i \quad \forall i \\ & x_i \leq 1 \forall i & y, \lambda_i \geq 0 \forall i \\ & x_i \geq 0 \, \forall i & \end{array}$$

In the above formulation, x_i represents the amount of company i that you have decided to find (1 would be 100%), p_i is the promised return from company i and r_i is the capital required for Company i. M is the amount of capital available to you, y and λ_is are the dual variables.

Let's look at an actual instance of this problem. Below are five companies that you could invest in:

Company	Required Capital ($M)	Promised Return ($M)
Company 1	4	3
Company 2	3	2
Company 3	10	7
Company 4	5	4
Company 5	5	3

The linear program corresponding to this problem is:

$$\begin{array}{rl} \text{maximize} & 3x_1 + 2x_2 + 7x_3 + 4x_4 + 3x_5 \\ \text{subject to:} & 4x_1 + 3x_2 + 10x_3 + 5x_4 + 5x_5 \leq 10 \\ & 0 \leq x_i \leq 1 \quad \forall i \in \{1, \cdots, 5\} \end{array}$$

The corresponding dual formulation would be:

$$\begin{aligned}
\text{minimize } & 10y + \sum_{i=1}^{5} \lambda_i \\
\text{subject to: } & 4y + \lambda_1 \geq 3 \\
& 3y + \lambda_2 \geq 2 \\
& 10y + \lambda_3 \geq 7 \\
& 5y + \lambda_4 \geq 4 \\
& 5y + \lambda_5 \geq 3 \\
& \lambda_i \geq 0 \quad \forall i \in \{1, \cdots, 5\}
\end{aligned}$$

Questions:

(a) One strategy (which turns out to be optimal) for solving this problem is to find the ratio $\frac{p_i}{r_i}$ for all the companies. First, invest in the company with the highest ratio with as much capital as you can (until you exhaust your capital or fulfill the company's capital requirement). If that doesn't exhaust your available capital, invest in the company with the second highest ratio with your remaining capital and continue this process until your funds are depleted.

We followed this strategy and discovered that the optimal solution to the primal problem was $(x_1, x_2, x_3, x_4, x_5) = (1, 0, \frac{1}{10}, 1, 0)$. Additionally, we are told that the optimal value of y is 0.7. What are the optimal values of the dual variables $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$?

(b) Give an economic interpretation of what the dual variable y represents. A sentence or two should suffice.

(c) Give an economic interpretation of what the dual variables λ_i represent. A sentence or two should suffice.