Discrete Time Markov Chains (DTMCs)

Discrete-Time Stochastic Process

A stochastic process $\{ X_t , t \in T \}$ is a collection of random variables.

T is the index set of the process: most often it represents time.
X_t is the state of the process at time t.

If T is countable, $\{ X_t , t \in T \}$ is a discrete-time stochastic process.

example: X_t is the amount of inventory on hand at the end of the day, for day t, $t \in T = [0,1,2,\ldots]$

The state space of the stochastic process is the set of possible values X_t can assume.

We will denote a discrete-time stochastic process by $X_0, X_1, X_2,\ldots$
Denote its state space by $E = \{0,\ldots.S\}$.
If $X_n=i$, we say the process is in state i at time n.
(We will use the notation X_n and X_t interchangeably.)

Overview of DTMC Lectures

Definition of a Discrete Time Markov Chain

- The Markov Property
  Example: gambler's ruin

Formulation of a Discrete Time Markov Chain

- Transition probabilities: matrix representation & transition diagrams
  Examples: taxi queuing system
  (s,S) inventory system
- Choosing the state space for a Markov Chain
  Example: weather forecasting

Overview of DTMC Lectures, continued

Characterizing the behavior of a Markov Chain over time

- N-step transition probabilities: definition and computation
  Example: brand switching/market share
- Unconditional and initial probability distributions
- Classification of states of a Markov Chain
- Steady state probabilities: proof of existence and computation
- Average costs and rewards in the long run
  Example: taxi queue

A simple Markov chain

EXAMPLE 1: Gambler's Ruin

Suppose I go to Sands Casino with $2 in my pocket. I'm going to play a series of blackjack games in which I bet $1 each game. With probability *p* I win the game, and I lose with probability *1-p.* I quit playing as soon as I have $4 or I run out of money.

After game *t* I have a certain amount of cash --call it X_t. My cash position after the next game is a random variable X_{t+1}.

The key property of this example is that my cash position after game *t+1* depends only on my cash position after game *t* (and how I did in game *t+1*). It does not depend on the history of outcomes before game *t*. This is the essence of the Markov Property.

Discrete-Time Markov Chains

MARKOV PROPERTY

A discrete time stochastic process $X_0, X_1, X_2,\dots$ is a Markov Chain if for all *t* and for every sequence of states $k_0, k_1, \dots, k_{t-1}$,i,j

$$P(X_{t+1}=j \mid X_t=i, X_{t-1}=k_{t-1},\dots, X_0=k_0) = P(X_{t+1}=j \mid X_t=i)$$

➡ the conditional distribution of the future state given the past states and the present state is independent of the past states and depends only on the present state.

STATIONARITY ASSUMPTION

$P(X_{t+1}=j \mid X_t=i)$ is independent of *t*

STATIONARY TRANSITION PROBABILITIES

$$p_{ij} \equiv P(X_{t+1}=j \mid X_t=i) \underset{\uparrow}{=} P(X_1=j \mid X_0=i)$$

stationarity assumption

Applications of Discrete-Time Markov Chains

- Queuing Systems
  - number of customers waiting in line at a bank at the beginning of each hour
  - number of machines waiting for repair at the end of each day
  - number of patients waiting for a lung transplant at beginning of each week
- Inventory Management
  - number of units in stock at beginning of each week
  - number of backordered units at end of each day
- Airline Overbooking
  - number of coach seats reserved at beginning of each day
- Finance
  - price of a given security when market closes each day

More Applications of Discrete-Time Markov Chains

- Epidemiology
  - number of foot-and-mouth infected cows at the beginning of each day
- Population Growth
  - size of US population at end of each year
- Genetics
  - genetic makeup of your descendants in each subsequent generation
- Workforce Planning
  - number of employees in a firm with each level of experience at the end of each year

Matrix Notation for Transition Probabilities

TRANSITION PROBABILITY MATRIX

$$P = \begin{bmatrix} p_{00} & p_{01} & \cdots & p_{0S} \\ p_{10} & p_{11} & \cdots & p_{1S} \\ \vdots & \vdots & & \vdots \\ p_{S0} & p_{S1} & \cdots & p_{SS} \end{bmatrix}$$

Transition Probabilities must satisfy

$$p_{ij} \geq 0 \text{ for all } i,j \in \{0,...,S\}$$
$$\sum_{j=0}^{S} p_{ij} = 1 \text{ for all } i \in \{0,...,S\}$$

Examples of Discrete Time Markov Chains

EXAMPLE 1 revisited

Define X_t to be the amount of money I have after playing game *t* (or the last game if game *t* was not played), for $T = \{0,1,2,....\}$.

State space of X_t : E = {\$0, \$1, \$2, \$3, \$4}

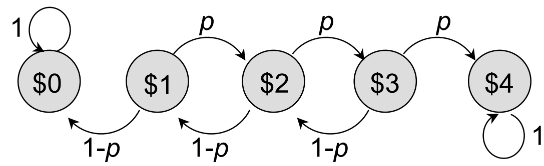
X_0 = \$2 is given

The distribution of X_{t+1} depends only on the past history only through the amount of money I have after game *t.*

➡ $X_0, X_1, X_2,\ldots$ is a Discrete-Time Markov Chain

Transition Diagram

TRANSITION DIAGRAM FOR EXAMPLE 1



TRANSITION MATRIX FOR EXAMPLE 1

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1-p & 0 & p & 0 & 0 \\ 0 & 1-p & 0 & p & 0 \\ 0 & 0 & 1-p & 0 & p \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Examples of Discrete Time Markov Chain

EXAMPLE 2: Taxi Queue

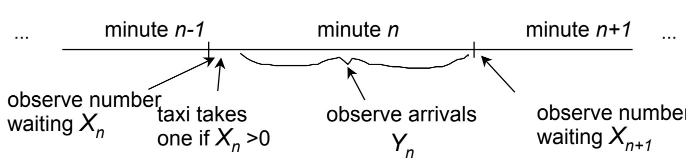
At SFO, people line up at the curb in hopes of catching a taxi home. Every minute, one taxi arrives, and if any customers are waiting, takes one customer away. If new arrivals find that there are N people waiting, they *balk* and take Supershuttle.

X_n = # waiting at the beginning of minute n

state space $E = \{0, 1, ..., N\}$

Y_n = # arrivals in minute n (iid)

$$P\{Y_n = k\} = \begin{cases} p_k & \text{for } k \geq 0 \\ 0 & \text{for } k < 0 \end{cases}$$



Queuing System, continued

Recursion $X_{n+1} = \begin{cases} \min(Y_n, N) & \text{if } X_n = 0 \\ \min(X_n - 1 + Y_n, N) & \text{if } X_n > 0 \end{cases}$

Want to calculate transition probabilities $p_{ij} = P\{X_{n+1} = j | X_n = i\}$

case 1: $i=0$

$$p_{0j} = P\{\min(Y_n, N) = j | X_n = 0\} = P\{\min(Y_n, N) = j\}$$

because X_n is independent of Y_n , so

$$= P\{\min(Y_n, N) = j\} = \begin{cases} p_j & \text{for } j<N \\ \sum_{k=N}^{\infty} p_k & \text{for } j=N \end{cases}$$

Queuing System, continued

case 2: $i>0$ $\quad p_{ij} = P\{X_{n+1} = j | X_n = i\} = P\{\min(X_n - 1 + Y_n, N) = j | X_n = i\}$

Using the definition of conditional probability,

$$p_{ij} = \frac{P\{\min(X_n - 1 + Y_n, N) = j, X_n = i\}}{P\{X_n = i\}} = \frac{P\{\min(i - 1 + Y_n, N) = j, X_n = i\}}{P\{X_n = i\}}$$

Since X_n is independent of Y_n ,

$$p_{ij} = \frac{P\{\min(i - 1 + Y_n, N) = j\} \cdot P\{X_n = i\}}{P\{X_n = i\}}$$

$$= P\{\min(i - 1 + Y_n, N) = j\}$$

$$= \begin{cases} P\{(i - 1 + Y_n) = j\} = p_{j+1-i} & \text{for } j<N \\ P\{(i - 1 + Y_n) \geq N\} = \sum_{k=N+1-i}^{\infty} p_k & \text{for } j = N \end{cases}$$

Queuing System transition matrix

$$P = \begin{bmatrix} p_0 & p_1 & p_2 & \cdots & \sum_{i=N}^{\infty} p_i \\ p_0 & p_1 & p_2 & \cdots & \sum_{i=N}^{\infty} p_i \\ 0 & p_0 & p_1 & \cdots & \sum_{i=N-1}^{\infty} p_i \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & p_0 & \sum_{i=1}^{\infty} p_i \end{bmatrix}$$

Discrete Approximation

Note: this is called a “discrete” queuing model because we have subdivided time into discrete units (minutes), and considered the arrivals within each unit of time as if they had arrived simultaneously. We will look at continuous queuing models, looking at arrivals as they appear, in the next part of the course.

Stationarity Assumption

Recall the stationarity assumption we made earlier:

$P(X_{t+1} = j \mid X_t = i)$ is independent of t

$$\Rightarrow P(X_{t+1} = j \mid X_t = i) = P(X_1 = j \mid X_0 = i)$$

In many real Markov Chains, this assumption does not hold. Consider our taxi queue. Most likely, the distribution of arrivals Y_n at time n would depend on the time of day, reflected in *n.* This would make the transition probabilities non-stationary.

The Markovian assumption allows the transition probabilities to be a function of time. Many interesting applications of Markov Chains are nonstationary.

Nonetheless, in this class, we make the assumption of stationarity.

A Nonstationary DTMC

The Polya Urn Model (Minh)

An urn contains *R* red and *G* green balls at step 0. At each step, we draw a ball at random from the urn and then return it with *k* additional balls of the same color.

After the *t*th step, the total number of balls is $R + G + tk$.

Let X_t be the number of red balls after then *t*th step, for t=0,1,.....

The state space of this Markov Chain is infinite: {0,1,2,.....}.

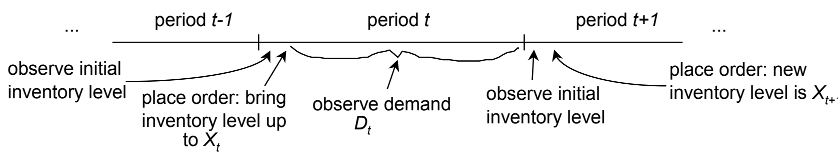
The transition probabilities are:

$$P(X_{t+1} = j \mid X_t = i) = \begin{cases} \dfrac{i}{R + G + tk} & \text{for } j = i + k \\ 1 - \dfrac{i}{R + G + tk} & \text{for } j = i \\ 0 & \text{otherwise} \end{cases}$$

Note that this is a nonstationary process.

Examples of Discrete Time Markov Chains

EXAMPLE 3: (s,S) Inventory System



(s,S) Inventory policy: If inventory level at beginning of a period is less than s, order up to S

(Assume instantaneous delivery of order.)

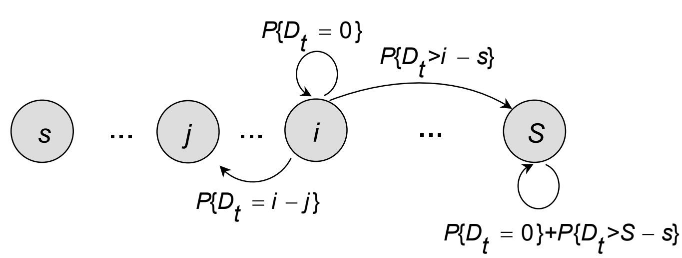
X_t = inventory level immediately after ordering decision in period t

state space $E = \{s, s+1, ..., S\}$

D_t = demand in period t (iid)

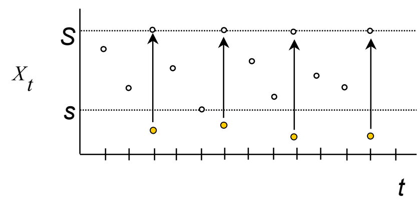
$$P\{D_t = k\} = \begin{cases} p_k & \text{for } k \geq 0 \\ 0 & \text{for } k < 0 \end{cases}$$

Transition Diagram for (s,S) Inventory System



$j \leq i < S$

(s,S) Inventory System, Cont'd



Recursio
n

$$X_{t+1} = \begin{cases} X_t - D_t & \text{if } X_t - D_t \geq s \\ S & \text{if } X_t - D_t < s \end{cases}$$

Transition Probabilities

$$p_{ij} = P\{X_{t+1} = j | X_t = i\} = \begin{cases} P\{D_t = i - j\} & \text{if } j < S \\ P\{D_t > i - s\} & \text{if } j = S,\ i < S \\ P\{D_t = 0\} + P\{D_t > S - s\} & \text{if } j = S,\ i = S \end{cases}$$

(s,S) Inventory system transition matrix

$$P = \begin{array}{c} \begin{matrix} s & s+1 & s+2 & & S \end{matrix} \\ \begin{bmatrix} p_0 & 0 & 0 & \cdots & \sum_{i=1}^{\infty} p_i \\ p_1 & p_0 & 0 & \cdots & \sum_{i=2}^{\infty} p_i \\ p_2 & p_1 & p_0 & \cdots & \sum_{i=3}^{\infty} p_i \\ \vdots & & & & \vdots \\ p_{S-s} & p_{S-s-1} & p_{S-s-2} & \cdots & p_0 + \sum_{i=S-s+1}^{\infty} p_i \end{bmatrix} \end{array}$$

Different Ways of Choosing State Space

Hillier and Lieberman use a different state space for this problem. In their version (p.803):

$\hat{X}_t$ = inventory level at the end of period t after demand is realized

Then $\hat{X}_t = X_t - D_t$

state space $E = \{0, 1, ..., S\}$

In this case either choice of state space works. However, in some examples, the choice of state space can be critical in determining whether the process is a Markov Chain.

Transforming a Process into a Markov Chain

EXAMPLE 4: Weather Forecasting (Ross)
Suppose that whether it rains or not depends on previous weather conditions through the last two days. If it has rained for the past two days, then it will rain tomorrow with probability .7; if it rained today but not yesterday, then it will rain tomorrow with probability .5; if it rained yesterday but not today, then it will rain tomorrow with probability .4; if it has not rained in the past two days, then it will rain tomorrow with probability .2.

If we let the state at time n represent whether it is raining at time n, then the process doesn't satisfy the Markov property. The conditional probability of rain at time $n+1$ given all past weather depends not only on the weather at time n but also on the weather at time $n-1$.

Transforming a Process into a Markov Chain

EXAMPLE 4, continued

Suppose instead we let the state of the process be today's *and* yesterday's weather. Define the state space as follows:

state 0	(rain, rain)	it rained today and yesterday
state 1	(sun, rain)	it rained today but not yesterday
state 2	(rain, sun)	it rained yesterday but not today
state 3	(sun, sun)	it rained neither today nor yesterday

The transition matrix for this Markov Chain is:

$$P = \begin{array}{c} \text{yesterday \& today} \end{array} \overset{\text{today \& tomorrow}}{\begin{bmatrix} .7 & 0 & .3 & 0 \\ .5 & 0 & .5 & 0 \\ 0 & .4 & 0 & .6 \\ 0 & .2 & 0 & .8 \end{bmatrix}}$$

N-Step Transition Probabilities

N-STEP TRANSITION PROBABILITIES

$$p_{ij}^{(n)} \equiv P(X_{t+n} = j \mid X_t = i) = P(X_n = j \mid X_0 = i)$$

= conditional probability that the stochastic process X_t will be in state *j* after exactly *n* steps, given that it starts in state *i.*

$P^{(n)} \equiv$ matrix of *n*-step probabilities (*ij*th element of $P^{(n)}$ is $p_{ij}^{(n)}$)

EXAMPLE 1 revisited

What's the probability that I will continue playing blackjack after 4 games?

$$P(0 < X_4 < 4 \mid X_0 = 2) = p_{21}^{(4)} + p_{22}^{(4)} + p_{23}^{(4)}$$

The Chapman-Kolmogorov Equations

To compute the n-step transition probability $p_{ij}^{(n)}$, pick any number of steps $m < n$. At time m, the process has got to be somewhere.

$$P(X_n = j \mid X_m = k)P(X_m = k \mid X_0 = i) = p_{kj}^{(n-m)} p_{ik}^{(m)}$$

is the conditional probability that, given a starting state of i, the process will be in state k after m steps and then in state j after another m - n steps. By summing over all possible states for time m, we get the Chapman-Kolmogorov Equations:

$$p_{ij}^{(n)} = \sum_{k=0}^{S} P(X_n = j \mid X_m = k)P(X_m = k \mid X_0 = i) = \sum_{k=0}^{S} p_{kj}^{(n-m)} p_{ik}^{(m)}$$

$$\text{for all } i,j \in \{0, \ldots, S\} \text{ and } 0 \le m \le n$$

Example: m = 1

$$p_{ij}^{(n)} = \sum_{k=0}^{S} p_{kj}^{(n-1)} p_{ik}$$

Example: m = n-1

$$p_{ij}^{(n)} = \sum_{k=0}^{S} p_{kj} p_{ik}^{(n-1)}$$

Computing N-Step Transition Probabilities

EXAMPLE: *n*=2

$$p_{ij}^{(2)} = \sum_{k=0}^{S} p_{ik} p_{kj} = \underbrace{\begin{pmatrix} p_{i0} & p_{i1} & \cdots & p_{iS} \end{pmatrix}}_{i\text{th row of } P} \underbrace{\begin{pmatrix} p_{0j} \\ p_{1j} \\ \vdots \\ p_{Sj} \end{pmatrix}}_{j\text{th column of } P}$$

ijth element of the 2-step $P^{(2)}$ transition matrix

= ijth element of the matrix $P \cdot P = P^2$

$$P^{(2)} = P \cdot P = P^2$$

COMPUTING N-STEP TRANSITION PROBABILITIES IN GENERAL

$$p_{ij}^{(n)} = \sum_{k=0}^{S} p_{kj}^{(n-1)} p_{ik}$$ = ijth element of the matrix $P^{(n-1)} \cdot P$

$$P^{(n)} = P^{(n-1)} \cdot P = P^{(n-2)} \cdot P \cdot P = \ldots = P \cdot P \cdots P = P^n$$

$$\boxed{P^{(n)} = P^n}$$

$P^{(0)} = P^0 = (S+1) \times (S+1)$ identity matrix, I

Matrix Multiplication Review

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$$

$$A = \begin{bmatrix} A_{00} & A_{01} & \cdots & A_{0n} \\ A_{10} & A_{11} & \cdots & A_{1n} \\ \vdots & \vdots & & \vdots \\ A_{n0} & A_{n1} & \cdots & A_{nn} \end{bmatrix} \qquad B = \begin{bmatrix} B_{00} & B_{01} & \cdots & B_{0n} \\ B_{10} & B_{11} & \cdots & B_{1n} \\ \vdots & \vdots & & \vdots \\ B_{n0} & B_{n1} & \cdots & B_{nn} \end{bmatrix}$$

Let $C = A\,B$. Then $C_{ij} = \underbrace{(A_{i0}, A_{i1}, \ldots, A_{in})}_{i\text{th row of } A} \begin{pmatrix} B_{0j} \\ B_{1j} \\ \vdots \\ B_{nj} \end{pmatrix} \leftarrow j\text{th column of } B$

$$= \sum_{k=0}^{n} A_{ik} B_{kj}$$

N-Step Transition Probabilities: Example

EXAMPLE 5: Coke vs. Pepsi (Winston)

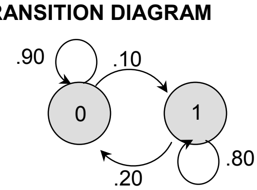
Given that a person’s last cola purchase was Coke, there is a 90% chance that her next cola purchase will also be Coke. If a person’s last cola purchase was Pepsi, there is an 80% chance that her next cola purchase will also be Pepsi.

Let X_t describe the type of cola she chose at her tth cola purchase, T={0,1,2,….}.

X_0 refers to the most recent cola purchase at present (unknown).

State space of X_t : E = {0,1} where 0 refers to Coke and 1 to Pepsi.

TRANSITION DIAGRAM



TRANSITION MATRIX

$$P = \begin{bmatrix} .90 & .10 \\ .20 & .80 \end{bmatrix}$$

N-Step Transition Probabilities: Example

EXAMPLE 5, continued

Given that a person is currently a Pepsi purchaser, what is the probability that she will purchase Coke two purchases from now?

$P(X_2 = 0 \mid X_0 = 1) = p_{10}^{(2)}$ = (1,0)th element of $P^{(2)}$

$$P^{(2)} = P^2 = P \cdot P = \begin{bmatrix} .90 & .10 \\ .20 & .80 \end{bmatrix} \begin{bmatrix} .90 & .10 \\ .20 & .80 \end{bmatrix} = \begin{bmatrix} .81 + 0.02 & .09 + .08 \\ .18 + .16 & .02 + .64 \end{bmatrix} = \begin{bmatrix} .83 & .17 \\ .34 & .66 \end{bmatrix}$$

Given that a person is currently a Coke drinker, what is the probability that she will purchase Pepsi three purchases from now?

$P(X_3 = 1 \mid X_0 = 0) = p_{01}^{(3)}$ = (0,1)th element of $P^{(3)}$

$$P^{(3)} = P^3 = P \cdot P^2 = \begin{bmatrix} .90 & .10 \\ .20 & .80 \end{bmatrix} \begin{bmatrix} .83 & .17 \\ .34 & .66 \end{bmatrix} = \begin{bmatrix} .781 & .219 \\ .438 & .562 \end{bmatrix}$$

Unconditional and Initial Probabilities

So far we've been talking about conditional probabilities:

$$p_{ij}^{(n)} \equiv P(X_{t+n} = j \mid X_t = i) = P(X_n = j \mid X_0 = i)$$

Suppose instead we're interested in the unconditional probability that the process is in a given state j at time n: .

$$P(X_n = j)$$

To compute these unconditional probabilities we need to know the initial state of the process, or at least have a probability distribution describing the initial state.

INITIAL PROBABILITY DISTRIBUTION

Let Q_j be the probability that the process is initially in state j, i.e.

$$Q_j \equiv P(X_0 = j) \qquad Q = (Q_0, Q_2, \ldots, Q_S)$$

Unconditional Probabilities

$$
\begin{aligned}
&P(X_n = j) \\
&= \sum_{i=0}^{S} P(X_0 = i)P(X_n = j \mid X_0 = i) \\
&= \sum_{i=0}^{S} (\text{prob. that initial state is } i) \times (\text{prob. of going from } i \text{ to } j \text{ in } n \text{ transitions}) \\
&= \sum_{i=0}^{S} Q_i p_{ij}^{(n)} = Q \cdot (\text{column } j \text{ of } P^{(n)})
\end{aligned}
$$

UNCONDITIONAL PROBABILITIES

The unconditional probability $P(X_n = j)$ that the process is in state *j* at time *n* satisfies

$$P(X_n = j) = \sum_{i=0}^{S} Q_i p_{ij}^{(n)}$$

Unconditional Probabilities: Example

EXAMPLE 5, continued

Assume each person makes one cola purchase per week. Suppose 60% of all people now drink Coke, and 40% drink Pepsi. What fraction of people will be drinking Coke three weeks from now?

Assume:

fraction of people that are initially drinking cola *j*

= probability that a given individual is initially drinking cola *j*

We are given $Q = (Q_0, Q_1) = (.6, .4)$

We want to find $P(X_3 = 0)$

$$P(X_3 = 0) = \sum_{i=0}^{1} Q_i p_{i0}^{(3)} = Q_0 p_{00}^{(3)} + Q_1 p_{10}^{(3)} = (.6)(.781) + (.4)(.438) = .6438$$

Long Run Behavior: Example

EXAMPLE 5, continued

What fraction of people will be drinking Coke twenty purchases from now? fifty purchases from now? one hundred?

$$P(X_n = 0) = \sum_{i=0}^{1} Q_i p_{i0}^{(n)} = Q_0 p_{00}^{(n)} + Q_1 p_{10}^{(n)} = Q \cdot (\text{0th column of } P^{(n)})$$

Using MATLAB, we can compute:

$$P^{(20)} = P^{20} = \begin{bmatrix} .6669 & .3331 \\ .6661 & .3339 \end{bmatrix} \quad P^{(50)} = \begin{bmatrix} .6667 & .3333 \\ .6667 & .3333 \end{bmatrix} \quad P^{(100)} = \begin{bmatrix} .6667 & .3333 \\ .6667 & .3333 \end{bmatrix}$$

Notice that $P^{(50)} = P^{(100)}$. In fact, for this example

$$P^{(n)} = \begin{bmatrix} .6667 & .3333 \\ .6667 & .3333 \end{bmatrix} \text{ for all } n \geq 30$$

Long Run Behavior: Example

EXAMPLE 5, continued

The fraction of people Coke drinking *n* purchases from now is:

$$P(X_n = 0) = Q_0 p_{00}^{(n)} + Q_1 p_{10}^{(n)}$$

$$P(X_{20} = 0) = (.6)(.6669) + (.4)(.6661) = .6666$$

Since $P^{(n)} = \begin{bmatrix} .6667 & .3333 \\ .6667 & .3333 \end{bmatrix}$ for all $n \geq 30$

$$P(X_n = 0) = (.6)(.6667) + (.4)(.6667) = .6667 \quad \text{for all } n \geq 30$$

For large n, $p_{00}^{(n)}$ and $p_{10}^{(n)}$ are nearly constant and approach .6667. Likewise, $P(X_n = 0)$ approaches .6667 for large *n*. This means that no matter what her initial preference, if you observe what she's drinking at some time in the distant future, there is a .6667 chance that it will be Coke. Similarly, there is a .3333 = 1- .6667 chance that she will be drinking Pepsi. The initial state becomes irrelevant in the long run.

Often we are *most* interested in what happens to a stochastic process in the long run, independent of initial conditions. In example 3, we saw the *n*-step transition probabilities begin to settle down after a large number of transitions regardless of the initial state. However, not all Markov Chains will exhibit this stabilizing behavior. We want to study the long-run properties of a Markov Chain. But to do this first we need to classify the states of a Markov Chain.

Classification of States of a Markov Chain

ACCESSIBLE

State *j* is accessible (or reachable) from state *i* if $p_{ij}^{(n)} > 0$ for some $n \geq 0$.

COMMUNICATE

States *i* and *j* communicate if *j* is accessible from *i* and *i* is accessible from *j*.

<u>Properties of communication:</u>

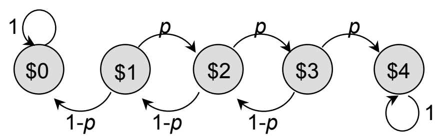
Reflexive: Every state communicates with itself (since $p_{ii}^{(0)} = 1$)

Symmetric: If *i* communicates with *j* then *j* communicates with *i*.

Transitive: If *i* communicates with *j* and *j* communicates with *k*, then *i* communicates with *k*.

Accessibility/Communication: Example

EXAMPLE 1: Gambler's Ruin



State 0 is accessible from 0,1,2,3.

State 1 is accessible from 1,2,3.

State 2 is accessible from 1,2,3.

State 3 is accessible from 1,2,3.

State 4 is accessible from 1,2,3,4.

State 1 communicates with 1,2,3.

State 2 communicates with 1,2,3.

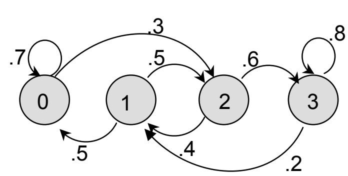
State 3 communicates with 1,2,3.

State 0 does not communicate with any other state.

State 4 does not communicate with any other state.

Accessibility/Communication: Example

EXAMPLE 4: Weather Prediction



State 0 is accessible from 0,1,2,3.

State 1 is accessible from 0,1,2,3.

State 2 is accessible from 0,1,2,3.

State 3 is accessible from 0,1,2,3.

State 0 communicates with 0,1,2,3.

State 1 communicates with 0,1,2,3.

State 2 communicates with 0,1,2,3.

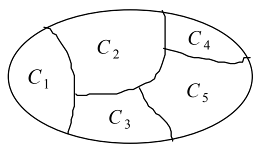
State 3 communicates with 0,1,2,3.

Classification of States

COMMUNICATING CLASSES

It follows from the properties of communication that the state space of a Markov Chain can be partitioned into one or more disjoint classes in which any two states that communicate belong to the same class. These classes are called communicating classes.

state space E



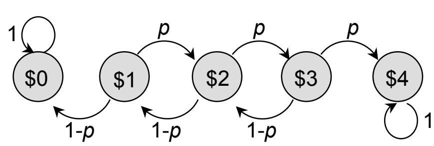
$$E = C_1 \cup C_2 \cup C_3 \cup C_4 \cup C_5$$

IRREDUCIBILITY

If the state space of a Markov Chain consists of only one communicating class, then the Markov Chain is said to be irreducible.

Communicating Classes: Example

EXAMPLE 1: Gambler's Ruin



Three communicating classes:

{1,2,3}, {0}, {4}

➡ not irreducible

State 1 communicates with 1,2,3.

State 2 communicates with 1,2,3.

State 3 communicates with 1,2,3.

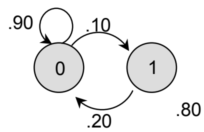
State 0 does not communicate with any other state.

State 4 does not communicate with any other state.

Irreducibility: Examples

EXAMPLE 5: Coke vs. Pepsi

$$P = \begin{bmatrix} .90 & .10 \\ .20 & .80 \end{bmatrix}$$



One communicating class: {0,1}

➡ irreducible

EXAMPLE 4: Weather Prediction

$$P = \begin{array}{cc} & \text{today \& tomorrow} \\ \text{yesterday \& today} & \begin{bmatrix} .7 & 0 & .3 & 0 \\ .5 & 0 & .5 & 0 \\ 0 & .4 & 0 & .6 \\ 0 & .2 & 0 & .8 \end{bmatrix} \end{array}$$

One communicating class: {0,1,2,3}

➡ irreducible

Classification of States

PERIODICITY

The period of state j is the largest integer k such that $p_{jj}^{(n)} = 0$ whenever n is not divisible by k.

State j is periodic (with period k) if k>1.

A state j is aperiodic if it is not periodic.

The period of state j = g.c.d. $\{n \geq 1 : p_{jj}^{(n)} > 0\}$

Intuition: If state j has period k, then starting in state j, you can only return to j after a multiple of k steps. The period k is the largest integer with this property.

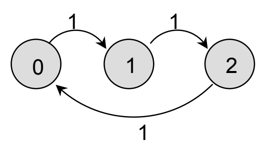
Periodicity: Examples

EXAMPLES: Greatest Common Denominator

g.c.d. $\{1, 3, 7, 8, 9, ...\} = 1$
g.c.d. $\{2, 6, 10, 14, 18, ...\} = 2$
g.c.d. $\{2, 16, 31, 49, ...\} = 1$

EXAMPLE: A Periodic Markov Chain

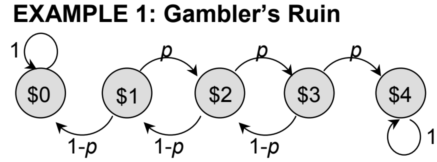
$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$



States 0,1 and 2 have period 3.

Periodicity: More Examples

EXAMPLE 1: Gambler's Ruin



States 1,2 and 3 have period 2.
States 0 and 4 are aperiodic.

EXAMPLE 4: Weather Prediction

$$P = \begin{array}{c} \\ \text{yesterday \& today} \end{array} \begin{array}{c} \text{today \& tomorrow} \\ \begin{bmatrix} .7 & 0 & .3 & 0 \\ .5 & 0 & .5 & 0 \\ 0 & .4 & 0 & .6 \\ 0 & .2 & 0 & .8 \end{bmatrix} \end{array}$$

All states are aperiodic.

Facts about Periodicity

It can be shown that periodicity is a class property: if state *i* has period *k*, then all states in *i*'s communicating class have period *k.*

If a Markov Chain is irreducible, then all states have the same period.

Note, if $p_{jj} > 0$, then state *j* is aperiodic.

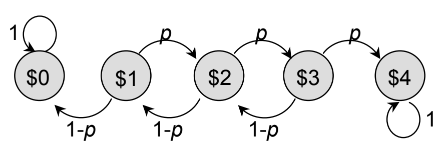
However, the converse is not true in general: it is possible to have an aperiodic state *j* for which $p_{jj} = 0$. (See previous example.)

Classification of States

ABSORBING STATE

State *j* is **absorbing** if once in state *j* the process will never leave state *j*, i.e., $p_{jj} = 1$.

EXAMPLE 1: Gambler's Ruin



States 0 and 4 are absorbing.

Recurrence and Transience: Introduction

We're often interested in determining which states of a stochastic process are visited most often in the long run.

In real world applications of Markov Chains, different states of the process can have different costs and benefits associated with them. For example, we might associate a very high cost, or penalty, with having zero inventory (a stockout.) Likewise, the cost of having the line go out the door in the bank is bad for customer service, and so would be considered more costly than having only 3 people in line.

For these reasons, we are interested in looking at whether, in the long run of a Markov Chain, certain states are likely to be visited. A state is called <u>recurrent</u> if, given that the Markov Chain starts in that state, it will return infinitely often to that state in the long run. A state is called <u>transient</u> if, given that the Markov Chain started in that state, there is a positive probability that it will stop visiting that state at some point (and so it will be visited only finitely many times in the long run.) In the next few lectures we are going to try to characterize recurrent and transient states.

First Passage Times

FIRST PASSAGE TIMES

Let T_{ij} be the <u>first time</u> the Markov Chain visits state j given that it starts in state i. Then T_{ij} is a random variable, which we call the first passage time from i to j. It can be written as $T_{ij} = \min\{ n \geq 1 : X_n = j \mid X_0 = i\}$

FIRST PASSAGE PROBABILITIES

Let $f_{ij}^{(n)}$ be the probability that, starting in i, the first transition into state j occurs at time n. Then $f_{ij}^{(n)}$ is called the first passage probability from i to j. It satisfies

$$f_{ij}^{(0)} = \begin{cases} 0 \text{ if } i \neq j \\ 1 \text{ if } i = j \end{cases}$$

$$\begin{aligned} f_{ij}^{(n)} &= P\{T_{ij} = n\} \\ &= P\{X_n = j, X_{n-1} \neq j, \ldots, X_1 \neq j \mid X_0 = i\} \end{aligned}$$

Classification of States

Let $\quad f_{ij} = \sum_{n=1}^{\infty} f_{ij}^{(n)}.$

Then f_{ij} is the probability that the process will *eventually* visit state *j* given that it starts in state *i*.

$$\Rightarrow \quad f_{ij} = P\{T_{ij} < \infty\}$$

RECURRENT AND TRANSIENT STATES

State *j* is **recurrent** if f_{jj} = 1 and **transient** if f_{jj} < 1.

An absorbing state is a special case of a recurrent state.

It's generally not easy to compute the $f_{ij}^{(n)}$'s , so it can be difficult to show whether f_{jj} = 1 or f_{jj} < 1. The following slides give some alternative ways of determining recurrence and transience.

Determining Recurrence & Transience

If state *i* is recurrent, then if the process starts in state *i,* it will eventually return to state *i* . As the process continues over an infinite number of time periods, the process will return to state *i* again and again and again ... infinitely often.

Suppose state *i* is transient. Each time the process enters state *i,* there's a positive probability, namely 1 – f_{ii} , that the process will never return to state *i*.

Let N_{ij} be the number of periods in which the process is in state *j* over infinite time, given that it started in state *i.*

State *i* is recurrent $\Leftrightarrow \quad E(N_{ii}) = \infty$

State *i* is transient $\Leftrightarrow \quad E(N_{ii}) < \infty$

Determining Recurrence & Transience

To calculate $E(N_{ij})$:

Define $B_n = \begin{cases} 1 \text{ if } X_n = j \\ 0 \text{ if } X_n \neq j \end{cases}$ ("Indicator" for process being in *j* in period *n*)

$$E(N_{ij}) = E(\sum_{n=1}^{\infty} B_n \mid X_0 = i) = \sum_{n=1}^{\infty} E(B_n \mid X_0 = i)$$

$$= \sum_{n=1}^{\infty} 1 \cdot P(X_n = j \mid X_0 = i) + 0 \cdot P(X_n \neq j \mid X_0 = i) = \sum_{n=1}^{\infty} p_{ij}^{(n)}$$

State *i* is recurrent $\Leftrightarrow$ $\sum_{n=1}^{\infty} p_{ii}^{(n)} = \infty$

State *i* is transient $\Leftrightarrow$ $\sum_{n=1}^{\infty} p_{ii}^{(n)} < \infty$

Facts about Recurrence & Transience

Not all states can be transient in a finite-state Markov Chain. Therefore, a finite-state Markov must have at least one recurrent state.

If state *i* is recurrent and state *i* communicates with state *j*, then state *j* is recurrent.

Proof: Since *i* and *j* communicate, there exist integers *k* and *m* such that

$$p_{ij}^{(k)} > 0 \text{ and } p_{ji}^{(m)} > 0 .$$

For any integer *n*, $p_{jj}^{(k+n+m)} \geq p_{ij}^{(k)} p_{ii}^{(n)} p_{ji}^{(m)}$.

Summing over *n*, we have $\sum_{n=1}^{\infty} p_{jj}^{(k+n+m)} \geq p_{ij}^{(k)} p_{ji}^{(m)} \sum_{n=1}^{\infty} p_{ii}^{(n)} = \infty$ since $p_{ij}^{(k)} p_{ji}^{(m)} > 0$.

It follows from the above fact that recurrence is a class property --either all states within a communicating class are recurrent or all states are transient. Therefore, all states in a finite, irreducible Markov Chain must be recurrent.

Recurrent and Transient States: Examples

EXAMPLE:

$$P = \begin{bmatrix} 0 & 0 & 1/2 & 1/2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

All states communicate & the chain is finite

➡ All states must be recurrent

EXAMPLE:

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 \\ 1/4 & 1/4 & 0 & 0 & 1/2 \end{bmatrix}$$

Communicating classes:{0,1} {2,3} {4}

$f_{00} = 1, f_{11} = 1, f_{22} = 1, f_{33} = 1$

➡ States 0,1,2, & 3 are recurrent

$f_{44} < 1$ ➡ State 4 is transient

Recurrent and Transient States: Examples

EXAMPLE 6: An Infinite Random Walk

Consider a Markov Chain with state space $E = \{0, \pm 1, \pm 2, \pm 3, ...\}$
and transition probabilities

$$p_{i,i+1} = p, \quad p_{i,i-1} = 1 - p, \quad p_{ij} = 0 \quad \text{for all } j \neq i-1, i+1.$$

Determine which states are recurrent and which are transient.

<u>**Solution:**</u>

All states communicate.

➡ Either all states are recurrent or all states are transient.

Therefore, we can select one state and try to establish whether it's recurrent or transient. Consider state 0. We want to determine whether

$$\sum_{n=1}^{\infty} p_{00}^{(n)}$$

is finite or infinite.

Recurrent and Transient States: Examples

Solution, continued

Cannot return to state 0 in an odd number of steps since its period is 2. Therefore:

$$p_{00}^{(2n+1)} = 0 \quad \text{for} \quad n = 0, 1, 2, \ldots .$$

To return to state 0 in an even number $2n$ of steps, we must move up exactly n times and down exactly n times. The probability of doing so is binomial:

$$p_{00}^{(2n)} = \binom{2n}{n} p^n (1-p)^n \quad \text{for} \quad n = 0, 1, 2, \ldots .$$

where $\binom{a}{b} = \dfrac{a!}{(a-b)!\,b!}$ for nonnegative integers a and b.

Now apply Stirling's formula: $n! \approx n^{n+1/2} e^{-n} \sqrt{2\pi}$

$$p_{00}^{(2n)} = \frac{(2n)!}{n!\,n!} p^n (1-p)^n = \frac{(2n)^{2n+1/2} e^{-2n} \sqrt{2\pi}}{(n^{n+1/2} e^{-n} \sqrt{2\pi})^2} p^n (1-p)^n = \frac{(4p(1-p))^n}{\sqrt{\pi n}}$$

Recurrent and Transient States: Examples

Solution, continued

$$\sum_{n=1}^{\infty} p_{00}^{(n)} = \sum_{n=1}^{\infty} p_{00}^{(2n)} = \sum_{n=1}^{\infty} \frac{(4p(1-p))^n}{\sqrt{\pi n}}$$ When will this sum converge?

Applying the ratio test with $a_n = \dfrac{(4p(1-p))^n}{\sqrt{\pi n}}$ we have

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(4p(1-p))^{n+1}}{\sqrt{\pi(n+1)}} \frac{\sqrt{\pi n}}{(4p(1-p))^n} = 4p(1-p) \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{(n+1)}} = 4p(1-p)$$

For $p \neq 1/2$, $0 \leq p \leq 1$, $4p(1-p) < 1$.

For $p = 1/2$, $4p(1-p) = 1$.

So $\sum_{n=1}^{\infty} p_{00}^{(n)} = \infty$ if and only if $p = 1/2$

➡ All states are recurrent if $p = 1/2$
and all states are transient if $p \neq 1/2$

Expected First Passage Times

EXPECTED FIRST PASSAGE TIMES

Let μ_{ij} be the expected number of transitions required for the first visit to state j, given that the process started in state i. Then μ_{ij} is called the expected first passage time from i to j. It satisfies

$$\mu_{ij} = E\{T_{ij}\} = \sum_{n=1}^{\infty} nP\{T_{ij} = n\} = \sum_{n=1}^{\infty} nf_{ij}^{(n)}$$

$$\mu_{ij} = E(T_{ij}) = \sum_{k} E(T_{ij} \mid \text{first transition is to } k)p_{ik}$$

$$= p_{ij} + \sum_{k \neq j}(1+\mu_{kj})p_{ik} = \underbrace{p_{ij} + \sum_{k \neq j} p_{ik}}_{=1} + \sum_{k \neq j}\mu_{kj}p_{ik} = 1 + \sum_{k \neq j}\mu_{kj}p_{ik}$$

goes directly to j in 1 step

goes first to $k \neq j$

A special case of the expected first passage time is the expected recurrence time μ_{jj} for state j. It is the expected number of periods between consecutive visits to state j.

Expected First Passage Times: Example

EXAMPLE:

$$P = \begin{bmatrix} .7 & .3 \\ .4 & .6 \end{bmatrix}$$

$$\mu_{ij} = 1 + \sum_{k \neq j}\mu_{kj}p_{ik}$$

$$\begin{aligned}
\mu_{00} &= 1 + \mu_{10}p_{01} = 1 + .3\mu_{10} \\
\mu_{10} &= 1 + \mu_{10}p_{11} = 1 + .6\mu_{10} \\
\mu_{01} &= 1 + \mu_{01}p_{00} = 1 + .7\mu_{01} \\
\mu_{11} &= 1 + \mu_{01}p_{10} = 1 + .4\mu_{01}
\end{aligned}
\quad \Rightarrow \quad
\begin{aligned}
\mu_{00} &= 1.75 \\
\mu_{10} &= 2.5 \\
\mu_{01} &= 3.33 \\
\mu_{11} &= 2.33
\end{aligned}$$

Classification of States

NULL RECURRENCE AND POSITIVE RECURRENCE

A recurrent state *i* is

null recurrent if $\mu_{ii} = E\{T_{ii}\} = \infty$

positive recurrent if $\mu_{ii} = E\{T_{ii}\} < \infty$

In words, a state *i* is null recurrent if it is expected to take infinitely long for the process to return to state *i*, given that is started there. State *i* is positive recurrent if its expected return time is finite.

We say a Markov Chain is positive recurrent if all of its states are positive recurrent. Likewise, a Markov Chain is null recurrent if all of its states are null recurrent.

Facts about Positive and Null Recurrence

A finite-state Markov Chain cannot have null recurrent states. Thus all recurrent states in a finite state Markov Chain are positive recurrent.

States in the same communicating class are either all transient, all positive recurrent or all null recurrent.

All states are positive recurrent in an irreducible finite state Markov Chain.

Establishing Positive Recurrence

Positive recurrence is a crucial property for a Markov Chain to be stable in the long run. When is a Markov Chain positive recurrent (i.e. when are all states positive recurrent?) We know that finite chains cannot have null recurrent states. But what about infinite chains? In general it can be very difficult to establish when a state in an infinite Markov Chain is positive or null recurrent. Even recurrence is difficult to verify for infinite chains.

The following theorem gives us a way to test for positive recurrence in irreducible chains. The nice thing about this theorem is that once we have used it to establish positive recurrence, we have already found the steady-state probability distribution for the process.

Establishing Positive Recurrence: Theorem

Suppose $X_0, X_1, X_2,\ldots$ is an irreducible discrete-time Markov Chain. Then $X_0, X_1, X_2,\ldots$ is positive recurrent if and only if there exists a solution

$$\pi = (\pi_0, \pi_1, \ldots, \pi_S)$$

to the system:

$$\pi_j = \sum_{i=0}^{S} \pi_i p_{ij} \quad \text{for} \quad j = 0, \ldots, S$$

$$\sum_{i=0}^{S} \pi_i = 1$$

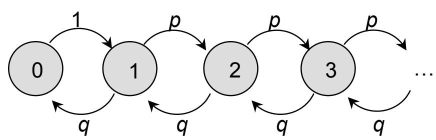
$$\pi_j \geq 0 \quad \text{for} \quad j = 0, 1, \ldots, S$$

Note: if a solution to the above system exists, it must be unique.

Remember that if you're testing for positive recurrence in a finite chain, you don't need this theorem. If the chain is irreducible, all states are automatically positive recurrent in that case.

Establishing Positive Recurrence: Example

EXAMPLE 7
Consider again an infinite random walk, this time on the nonnegative integers {0,1,….}. Given the process is in state i>0 in any period, it will be in state i+1 with probability p (0<p<1) in the next period and in state i-1 with probability q=1-p. If it is in state 0 this period, it will be in state 1 next period with probability 1. The transition diagram for this Markov Chain is:



And the transition matrix is:

$$P = \begin{bmatrix} 0 & 1 & 0 & . & . & . \\ q & 0 & p & . & . & . \\ 0 & q & 0 & p & . & . \\ 0 & 0 & q & 0 & p & . \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

Establishing Positive Recurrence: Example

EXAMPLE 7, continued
Since p and q are assumed to be positive, this is an irreducible Markov Chain. However, since it's an infinite chain we cannot assume that it is positive recurrent. To determine whether it is positive recurrent, we try to find a solution to:

$$\pi_j = \sum_{i=0}^{\infty} \pi_i p_{ij} \quad \text{for} \quad j = 0,1,...$$
$$\sum_{i=0}^{\infty} \pi_i = 1$$
$$\pi_j \geq 0 \quad \text{for} \quad j = 0,1,.....$$

For this example, these equations can be written as

$$\pi_0 = q\pi_1$$
$$\pi_1 = \pi_0 + q\pi_2$$
$$\pi_j = p\pi_{j-1} + q\pi_{j+1} \quad \text{for} \quad j = 2,3,....$$

$$\sum_{i=0}^{\infty} \pi_i = 1$$
$$\pi_j \geq 0 \quad \text{for} \quad j = 0,1,...$$

Establishing Positive Recurrence: Example

EXAMPLE 7, continued
We can solve the first group of equations recursively:

$$\pi_1 = \pi_0/q$$
$$\pi_2 = (\pi_1 - \pi_0)/q = p\pi_0/q^2$$
$$\pi_3 = (\pi_2 - p\pi_1)/q = (p\pi_0/q^2 - p\pi_0/q)/q = p^2\pi_0/q^3$$
....
$$\pi_j = p\pi_{j-1}/q = p^{j-1}\pi_0/q^j$$

Now we use the equation $\sum_{i=0}^{\infty} \pi_i = 1$

$$\sum_{i=0}^{\infty} \pi_i = \pi_0 + \sum_{i=1}^{\infty}\left(\frac{p^{i-1}}{q^i}\right)\pi_0 = \pi_0\left(1 + \sum_{i=1}^{\infty}\left(\frac{p^{i-1}}{q^i}\right)\right) = 1$$

So $\pi_0 = 1\Big/\left(1 + \sum_{i=1}^{\infty}\left(\frac{p^{i-1}}{q^i}\right)\right)$

Establishing Positive Recurrence: Example

EXAMPLE 7, continued

The sum $\sum_{i=1}^{\infty}\left(\frac{p^{i-1}}{q^i}\right) = \frac{1}{q}\sum_{i=0}^{\infty}\left(\frac{p}{q}\right)^i$ converges to $\frac{1}{q-p}$ if and only if $p<q$.

If $p < q$, $\pi_0 = \frac{1}{(1 + 1/(q-p))} = \frac{q-p}{(q-p+1)} = \frac{q-p}{2q}$

and $\pi_j = \frac{p^{j-1}}{q^j}\pi_0 = \frac{(q-p)p^{j-1}}{2q^{j+1}}$

If $p \geq q$, then $\pi_0 = 0$ and $\pi_j = p^{j-1}\pi_0 / q^j = 0$ In this case $\sum_{i=0}^{\infty}\pi_i = 0 \neq 1$

We have shown that the chain is positive recurrent if and only if *p<q.*

Big Theorem for Aperiodic Markov Chains

Theorem 1

If $X_0, X_1, X_2, \ldots$ is an irreducible, positive recurrent, aperiodic Markov Chain then $\lim_{n \to \infty} p_{ij}^{(n)}$ exists and is independent of *i.*

Define $\pi_j = \lim_{n \to \infty} p_{ij}^{(n)}$. Then the π_j's are the unique nonnegative solution to the system: $\pi_j = \sum_{i=0}^{S} \pi_i p_{ij}$ for $j = 0, \ldots, S$

$$\sum_{i=0}^{S} \pi_i = 1$$

Steady State Probabilities

$\pi = (\pi_0, \pi_1, \ldots, \pi_S)$ is called the vector of **steady state probabilities.**

Intuition for Big Theorem

Since $\pi_j = \lim_{n \to \infty} p_{ij}^{(n)}$ the limit of the *n*-step transition matrix is:

$$\lim_{n \to \infty} P^{(n)} = \begin{bmatrix} \pi_0 & \pi_1 & \cdots & \pi_S \\ \pi_0 & \pi_1 & \cdots & \pi_S \\ \vdots & \vdots & & \vdots \\ \pi_0 & \pi_1 & \cdots & \pi_S \end{bmatrix}$$

all the rows are the same!

The intuition behind the big theorem is that if the Markov Chain is irreducible, positive recurrent and aperiodic, the probability of being in state *j* after a large number of transitions is <u>independent of which state you started out in</u>.

Thus, π_j is the probability of being in state *j* in some randomly selected period in the distant future.

Steady State Equations

Steady State Equations

The S+2 equations $\pi_j = \sum_{i=0}^{S} \pi_i p_{ij}$ for $j = 0,\ldots,S$

$$\sum_{i=0}^{S} \pi_i = 1$$

are called the **steady state equations.**

The first S+1 equations follow from the Chapman-Kolmogorov Equations:

$$p_{kj}^{(n+1)} = \sum_{i=0}^{S} p_{ki}^{(n)} p_{ij}$$

as $n \to \infty$ $\downarrow$ $\downarrow$

$$\pi_j = \sum_{i=0}^{S} \pi_i p_{ij}$$

The last equation $\sum_{i=0}^{S} \pi_i = 1$ follows because $\pi = (\pi_0, \pi_1, \ldots, \pi_S)$ are probabilities.

Steady State Probabilities & Expected Recurrence Time

Remember that μ_{jj} represents the expected time between consecutive visits to state j.

It can be shown that $\pi_j = \frac{1}{\mu_{jj}}$. Thus π_j can also be interpreted as the long run proportion of time that the Markov Chain will be in state j.

For positive recurrent states j, $\mu_{jj} < \infty$. Thus $\pi_j > 0$.

Vector Notation for Steady State Equations

$$\pi_j = \sum_{i=0}^{S} \pi_i p_{ij} = (\pi_0, \pi_1, \ldots, \pi_S)\begin{pmatrix} p_{0j} \\ p_{1j} \\ \vdots \\ p_{Sj} \end{pmatrix} \leftarrow j\text{th column of } P$$

$$\sum_{i=0}^{S} \pi_i = (\pi_0, \pi_1, \ldots, \pi_S)\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \leftarrow \text{vector of ones is denoted by } e$$

So the steady state equations can be rewritten in vector form as

$$\boxed{\begin{aligned} \pi &= \pi P \\ \pi e &= 1 \end{aligned}}$$

Solving for the Steady State Probabilities

The system $\boxed{\begin{aligned} \pi &= \pi P \\ \pi e &= 1 \end{aligned}}$ consists of S+2 equations and S+1 unknowns. Therefore at least one equation is redundant and can be deleted. Note, however, that the equation $\pi e = 1$ is not the redundant one, since if we deleted it, $\pi = (0, 0, \ldots, 0)$ would be a solution, and it is not a probability distribution. Therefore, one of the S+1 equations in the system $\pi = \pi P$ is redundant: we can delete whichever one we like.

Solving for the Steady State Probabilities: Examples

EXAMPLE 5: Coke vs. Pepsi (Winston)

$$P = \begin{bmatrix} .90 & .10 \\ .20 & .80 \end{bmatrix}$$

$$\boxed{\begin{aligned} \pi &= \pi P \\ \pi e &= 1 \end{aligned}} \Rightarrow \begin{aligned} \pi_0 &= \pi_0 p_{00} + \pi_1 p_{10} \\ \pi_1 &= \pi_0 p_{01} + \pi_1 p_{11} \\ \pi_0 + \pi_1 &= 1 \end{aligned} \Rightarrow \begin{aligned} \pi_0 &= .9\pi_0 + .2\pi_1 \\ \pi_1 &= .1\pi_0 + .8\pi_1 \\ \pi_0 + \pi_1 &= 1 \end{aligned}$$

Solution: $\pi_0 = .66\overline{6}$, $\pi_1 = .33\overline{3}$

Solving for the Steady State Probabilities: Examples

EXAMPLE 4 revisited

$$P = \begin{bmatrix} .7 & 0 & .3 & 0 \\ .5 & 0 & .5 & 0 \\ 0 & .4 & 0 & .6 \\ 0 & .2 & 0 & .8 \end{bmatrix}$$

$$\boxed{\begin{aligned} \pi &= \pi P \\ \pi e &= 1 \end{aligned}} \Rightarrow \begin{aligned} \pi_0 &= \pi_0 p_{00} + \pi_1 p_{10} + \pi_2 p_{20} + \pi_3 p_{30} &&= .7\pi_0 + .5\pi_1 \\ \pi_1 &= \pi_0 p_{01} + \pi_1 p_{11} + \pi_2 p_{21} + \pi_3 p_{31} &&= .4\pi_2 + .2\pi_3 \\ \pi_2 &= \pi_0 p_{02} + \pi_1 p_{12} + \pi_2 p_{22} + \pi_3 p_{32} &&= .3\pi_0 + .5\pi_1 \\ \pi_3 &= \pi_0 p_{03} + \pi_1 p_{13} + \pi_2 p_{23} + \pi_3 p_{33} &&= .6\pi_2 + .8\pi_3 \\ \pi_0 + \pi_1 + \pi_2 + \pi_3 &= 1 \end{aligned}$$

Solution: $\pi_0 = .25$, $\pi_1 = \pi_2 = .15$, $\pi_3 = .45$

Solving for Steady State Probabilities: Examples

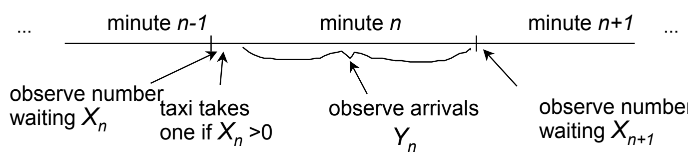
EXAMPLE 2 revisited
Consider our Discrete Queuing Model example again.

X_n = # waiting at the beginning of minute n

state space $E = \{0, 1, ..., N\}$

Y_n = # arrivals in minute n (iid)

$$P\{Y_n = k\} = \begin{cases} p_k & \text{for } k \geq 0 \\ 0 & \text{for } k < 0 \end{cases}$$



Solving for Steady State Probabilities: Examples

EXAMPLE 2, continued

Transition Probabilities $\quad p_{ij} = P\{X_{n+1} = j | X_n = i\}$

We showed
$$p_{0j} = P\{\min(Y_n, N) = j\} = \begin{cases} p_j & \text{for } j<N \\ \sum_{k=N}^{\infty} p_k & \text{for } j=N \end{cases}$$

and for i>0,
$$p_{ij} = \begin{cases} P\{(i-1+Y_n) = j\} = p_{j+1-i} & \text{for } j<N \\ P\{(i-1+Y_n) \geq N\} = \sum_{k=N+1-i}^{\infty} p_k & \text{for } j = N \end{cases}$$

For the case N = 2,
$$P = \begin{bmatrix} p_0 & p_1 & \sum_{k=2}^{\infty} p_k \\ p_0 & p_1 & \sum_{k=2}^{\infty} p_k \\ 0 & p_0 & \sum_{k=1}^{\infty} p_k \end{bmatrix} = \begin{bmatrix} p_0 & p_1 & 1-p_0-p_1 \\ p_0 & p_1 & 1-p_0-p_1 \\ 0 & p_0 & 1-p_0 \end{bmatrix}$$

Solving for Steady State Probabilities: Examples

EXAMPLE 2, continued

$$P = \begin{bmatrix} p_0 & p_1 & 1-p_0-p_1 \\ p_0 & p_1 & 1-p_0-p_1 \\ 0 & p_0 & 1-p_0 \end{bmatrix}$$

As long as $p_0 > 0$ and $0 < p_0 + p_1 < 1$, this Markov Chain is irreducible. Since it is also finite, it must be positive recurrent. Therefore, the steady state probabilities exist. Solving the steady state equations yields:

$$\pi_0 = \frac{(p_0)^2}{1-p_1}$$
$$\pi_1 = \frac{p_0(1-p_0)}{1-p_1}$$
$$\pi_2 = \frac{1-p_0-p_1}{1-p_1}$$

Solving for the Steady State Probabilities with MATLAB

We can rewrite the system $\boxed{\begin{array}{l} \pi = \pi P \\ \pi e = 1 \end{array}}$ as $\boxed{\begin{array}{l} \pi(I-P) = 0 \\ \pi e = 1 \end{array}}$ where I is the (S+1) x (S+1) identity matrix.

More explicitly,

This part is I-P with the last column deleted.

$$(\pi_0, \pi_1, \ldots, \pi_S) \begin{bmatrix} 1-p_{00} & -p_{01} & \cdots & -p_{0,S-1} & 1 \\ -p_{10} & 1-p_{11} & \cdots & -p_{1,S-1} & 1 \\ \vdots & & \ddots & & \vdots \\ -p_{S0} & -p_{S1} & \cdots & -p_{S,S-1} & 1 \end{bmatrix} = (\underbrace{0, 0, \ldots, 0}_{(S \text{ zeros})}, 1)$$

Call this matrix *A*.

A has an inverse, A^{-1}. Solve for π with Matlab using:

$$\boxed{\pi = (0, 0, \ldots, 0, 1) A^{-1}}$$

The Periodic Case

Theorem 1 applies to irreducible, positive recurrent, aperiodic Markov Chains. This result excludes periodic Markov Chains. Why?

Consider the following example:

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

This chain is irreducible and positive recurrent since it is finite. However, both states have period 2.

Notice:
$$p_{00}^{(n)} = 1 \text{ if } n \text{ is even;}$$
$$p_{00}^{(n)} = 0 \text{ if } n \text{ is odd.}$$

Since $p_{00}^{(n)}$ oscillates between 0 and 1 as $n \to \infty$

$$\lim_{n \to \infty} p_{00}^{(n)}$$

does not exist, so Theorem 1 doesn't hold.

However, we can show that

$$\lim_{n \to \infty} \frac{1}{n+1} \sum_{k=0}^{n} p_{ij}^{(k)}$$

does exist, and can be interpreted as the long run *average* probability of being in state *j.*

Big Theorem for the Periodic Case

Theorem 2

If $X_0, X_1, X_2, \ldots$ is an irreducible, positive recurrent, *possibly periodic* Markov Chain then

$$\lim_{n \to \infty} \frac{1}{n+1} \sum_{k=0}^{n} p_{ij}^{(k)}$$

exists and is independent of *i*.

Define $\pi_j = \lim_{n \to \infty} \frac{1}{n+1} \sum_{k=0}^{n} p_{ij}^{(k)}$.

Then the π_j's are the unique nonnegative solution to the system:

$$\pi_j = \sum_{i=0}^{S} \pi_i p_{ij} \quad \text{for} \quad j = 0, \ldots, S$$
$$\sum_{i=0}^{S} \pi_i = 1$$

Distinction between Periodic and Aperiodic Cases

In the periodic as well as the aperiodic cases, $\pi_j = 1/\mu_{jj}$. Thus even in the periodic case π_j can still be interpreted as the long run proportion of time that the Markov Chain will be in state *j*.

However, in the periodic case, π_j does not represent the long run probability of being in state *j*. Instead it represents the long run *average* probability of being in state *j.*

Interpretation as Stationary Probabilities

The steady state probability distribution π of a Markov Chain is also often referred to as its **stationary probability distribution**. The logic behind this terminology is as follows.

Recall the definition of the initial probability distribution *Q:*

$$Q_j \equiv P(X_0 = j) \qquad Q = (Q_0, Q_2, \ldots, Q_S)$$

Now suppose that $Q = \pi$. Observe that for every period n=1,2,….

$$P(X_n = j) = \sum_{i=0}^{S} Q_i p_{ij}^{(n)} = \sum_{i=0}^{S} \pi_i p_{ij}^{(n)} = \pi_j$$

What we have shown is that if the initial state of the process has distribution π, then unconditional distribution of the state in <u>every</u> subsequent period has distribution π. In other words, the probability of being in state *j* is <u>always</u> π_j.

Costs and Rewards in Discrete-Time Markov Chains

Perhaps the most important use of the steady state distribution of a Markov Chain is in computing the expected costs or rewards that are incurred as the process evolves.

For example, suppose you run a bank, and you're trying to decide how many tellers should be available. For any fixed number of tellers, we can model the line of customers as a Markov Chain. Presumably, as bank manager, you attach some importance to customer satisfaction. To model its importance, create a cost function that represents the negative effects of a long line on customer satisfaction. This function assigns a certain penalty to each possible length of the line. You would be interested in determining, for a fixed number of tellers, what the expected average cost per unit time is. In the next section, we'll learn the tools to compute this expectation. Using this information, you can then make a decision about how many tellers to have available by trading off the cost of additional tellers and the expected increase in customer satisfaction.

Long Run Expected Average Cost

Suppose there is a cost (or reward) $C(j)$ incurred when the process is in state j. Let

$$C = (C(0), C(1), \ldots, C(S))$$

The expected average cost incurred per period for first n periods of the process is

$$E\left[\frac{1}{n}\sum_{t=0}^{n-1} C(X_t)\right]$$

LONG RUN EXPECTED AVERAGE COST PER UNIT TIME

If $X_0, X_1, X_2, \ldots$ is an irreducible, positive recurrent Markov Chain then the **long run expected average cost per period** is

$$\lim_{n\to\infty} E\left[\frac{1}{n}\sum_{t=0}^{n-1} C(X_t)\right] = \sum_{j=0}^{S} \pi_j C(j) = \pi C$$

Long Run Expected Average Cost: Example

EXAMPLE 5 again
Recall our brand switching problem concerning Coke and Pepsi. The Coca-Cola company estimates that each time an individual purchases Coke, they earn \$0.25 in profits. Naturally they earn nothing when Pepsi is purchased. What is the long run expected average profit earned from each individual per cola purchase?

Solution: The long run expected average profit is $\pi C = \sum_{j=0}^{S} \pi_j C(j)$

where the vector $C = (.25, 0)$ now refers to profit rather than cost.

Recall that we found $\pi_0 = .66\overline{6}$, $\pi_1 = .33\overline{3}$.

Thus the long run expected average profit that Coke earns per individual per cola purchase is

$$\pi C = .25 \cdot .66\overline{6} + 0 \cdot .33\overline{3} = .16\overline{6}$$
$$= \$.167$$

Complex Cost Functions

In the preceding example(s) the cost that was incurred in each period depended only on the state in that period. In some situations the cost or reward in period t might depend not only on the state X_t but on another random variable Y_t associated with that period. Thus the cost (or reward) can be written instead as $C(X_t, Y_t)$.

EXAMPLE 2 (Taxi Queuing Model)
Consider, for example, our queuing problem. Suppose that the taxi company which services that particular curb estimates that is costs them \$20 each time they lose a customer to Super Shuttle.

In this problem, the cost that the taxi company faces in a given minute t depends not only on the state of the system (i.e., the number of customers at the curb), but also on the random variable Y_t, the number of potential customers that arrive during the minute t. Recall that X_t represents the number of people standing at the curb at the beginning of minute t.We would like to determine the long run expected cost (of losing customers) per minute to the taxi company.

Complex Cost Functions: Example

EXAMPLE 2, continued

To determine the cost in minute *t,* we first observe that if $X_t = 0$ then the number of people who take Supershuttle in period *t* is:

$$\max \{Y_t - N, 0\} = (Y_t - N)^+$$

where $x^+ = \max\{x, 0\}$, $x^- = \max\{-x, 0\}$

If instead $X_t > 0$ then the number of people who take Supershuttle in period *t* is

$$\max \{X_t - 1 + Y_t - N, 0\} = (X_t - 1 + Y_t - N)^+$$

Therefore depending on whether $X_t = 0$ or $X_t > 0$, we can write the cost in period *t* as

$$C(0, Y_t) = 20 \quad (Y_t - N)^+$$

$$C(j, Y_t) = 20 \quad (j - 1 + Y_t - N)^+$$

Complex Cost Functions: Example

EXAMPLE 2, continued

Let's suppose N=2. What is the long run expected average cost to the taxi company per minute?

<u>Solution</u>:
In this situation it can be shown that since the Y_t are independent and identically distributed, the long run expected average cost per minute is

$$\lim_{n \to \infty} E\left[\frac{1}{n}\sum_{t=0}^{n-1} C(X_t, Y_t)\right] = \sum_{j=0}^{S} \pi_j E[C(j, Y_t)] = \pi E[C]$$

In this problem, our cost vector is then

$$\begin{aligned} C &= (C(0, Y_t), C(1, Y_t), C(2, Y_t)) \\ &= (20 \ \max\{Y_t - 2, 0\},\ 20 \ \max\{Y_t - 2, 0\},\ 20 \ \max\{Y_t - 1, 0\}) \\ &= (20 \ (Y_t - 2)^+,\ 20 \ (Y_t - 2)^+,\ 20 \ (Y_t - 1)^+) \end{aligned}$$

To compute the long run cost per unit time for the taxi company, we must first compute the vector of expected costs $E[C]$.

Complex Cost Functions: Example

EXAMPLE 2, continued

The expected cost vector in period *t* is: $E[C] = 20\ (E[(Y_t - 2)^+],\ E[(Y_t - 2)^+],\ E[(Y_t - 1)^+])$

Now notice that $E[(Y_t - i)^+] = \sum_{k=0}^{\infty} p_k (k - i)^+ = \sum_{k=i+1}^{\infty} p_k (k - i)$

Letting $\mu = E[Y_t]$ and using the above, we have

$$E[(Y_t - 2)^+] = \sum_{k=3}^{\infty} p_k (k - 2) = \sum_{k=3}^{\infty} p_k k - 2\sum_{k=3}^{\infty} p_k$$

$$= (\mu - p_1 - 2p_2) - 2(1 - p_0 - p_1 - p_2) = \mu + p_1 + 2p_0 - 2$$

$$E[(Y_t - 1)^+] = \sum_{k=2}^{\infty} p_k (k - 1) = \sum_{k=2}^{\infty} p_k k - \sum_{k=2}^{\infty} p_k$$

$$= \mu - p_1 - (1 - p_0 - p_1) = \mu + p_0 - 1$$

So our expected cost vector can be written as:

$$E[C] = 20\ (\mu + 2p_0 + p_1 - 2,\ \mu + 2p_0 + p_1 - 2,\ \mu + p_0 - 1)$$

Complex Cost Functions: Example

$$E[C] = 20\ (\mu + 2p_0 + p_1 - 2,\ \mu + 2p_0 + p_1 - 2,\ \mu + p_0 - 1)$$

We have previously shown that $\pi_0 = \frac{(p_0)^2}{1 - p_1},\ \pi_1 = \frac{p_0(1 - p_0)}{1 - p_1},\ \pi_2 = \frac{1 - p_0 - p_1}{1 - p_1}.$

Therefore, the long run expected cost per unit time for the taxi company is

$$\sum_{j=0}^{S} \pi_j E[C(j, Y_t)] = \pi E[C]$$

$$= \frac{20}{1 - p_1}\Big[(p_0)^2(\mu + 2p_0 + p_1 - 2)$$

$$+ p_0(1 - p_0)(\mu + 2p_0 + p_1 - 2)$$

$$+ (1 - p_0 - p_1)(\mu + p_0 - 1))\Big]$$

$$= \frac{20}{1 - p_1}\left[(p_0)^2 + \mu - \mu p_1 - 1 + p_1\right]$$

$$= 20\ (\mu - 1) + 20\ \frac{(p_0)^2}{1 - p_1}$$

Long Run Expected Average Cost for Complex Cost Functions

LONG RUN EXPECTED AVERAGE COST PER UNIT TIME FOR COMPLEX COST FUNCTIONS

Suppose $X_0, X_1, X_2, \ldots$ is an irreducible, positive recurrent Markov Chain and $Y_0, Y_1, Y_2, \ldots$ is a sequence of independent, identically distributed random variables and Y_t is independent of $X_0, X_1, \ldots, X_t$. Suppose the cost in period t is $C(X_t, Y_t)$. Define the vector

$$E[C] = (E[C(0, Y_t)], E[C(1, Y_t)], \ldots, E[C(S, Y_t)])$$

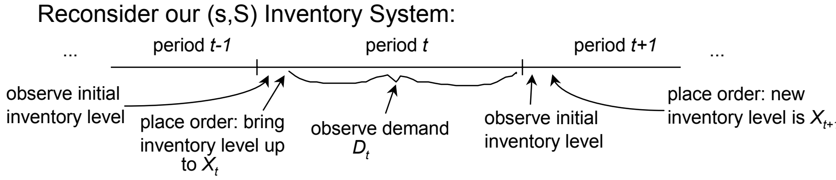
Then the **long run expected average cost per period** is given by

$$\lim_{n\to\infty} E\left[\frac{1}{n}\sum_{t=0}^{n-1} C(X_t, Y_t)\right] = \sum_{j=0}^{S} \pi_j E[C(j, Y_t)] = \pi E[C]$$

Complex Cost Functions: Example

EXAMPLE 3 revisited

Reconsider our (s,S) Inventory System:



(*s*,*S*) Inventory policy: If inventory level at beginning of a period is less than s, order up to S

(Assume instantaneous delivery of order.)

X_t = inventory level immediately after ordering decision in period t

state space $E = \{s, s+1, \ldots, S\}$

D_t = demand in period t (iid)

$$P\{D_t = k\} = \begin{cases} p_k & \text{for } k \geq 0 \\ 0 & \text{for } k < 0 \end{cases}$$

Complex Cost Functions: Example

EXAMPLE 3, continued

Transition Probabilities

$$p_{ij} = P\{X_{t+1} = j | X_t = i\} = \begin{cases} P\{D_t = i-j\} & \text{if } j < S \\ P\{D_t > i-s\} & \text{if } j=S,\ i < S \\ P\{D_t = 0\} + P\{D_t > S-s\} & \text{if } j=S,\ i = S \end{cases}$$

For the case (s,S) = (1,3),

$$P = \begin{bmatrix} p_0 & 0 & \sum_{k=1}^{\infty} p_k \\ p_1 & p_0 & \sum_{k=2}^{\infty} p_k \\ p_2 & p_1 & p_0 + \sum_{k=3}^{\infty} p_k \end{bmatrix} = \begin{bmatrix} p_0 & 0 & 1-p_0 \\ p_1 & p_0 & 1-p_0-p_1 \\ p_2 & p_1 & 1-p_1-p_2 \end{bmatrix}$$

Assume $p_0 = p_2 = .2,\ p_1 = .6$

So $P = \begin{bmatrix} .2 & 0 & .8 \\ .6 & .2 & .2 \\ .2 & .6 & .2 \end{bmatrix}$

This Markov Chain is irreducible. Since it is also finite, it must be positive recurrent. Therefore, the steady state probabilities exist. Solving the steady state equations yields…. $\pi_1 = \frac{13}{41},\ \pi_2 = \frac{12}{41},\ \pi_3 = \frac{16}{41}$

Complex Cost Functions: Example

EXAMPLE 3, continued

Suppose that the following costs are incurred in each period:

- \$10 stockout penalty per unit of backlogged demand
- \$1 inventory holding cost per unit leftover at the end of the period

Then the cost in period *t* can be expressed as:

$$C(X_t, D_t) = 10\ (D_t - X_t)^+ + 1(X_t - D_t)^+$$

where $x^+ = \max\{x, 0\}$, $x^- = \max\{-x, 0\}$

To compute the long run cost per period, we must first compute the vector

$$\begin{aligned} E[C] &= (E[C(1, D_t)],\ E[C(2, D_t)],\ E[C(3, D_t)]) \\ &= [10\ E(D_t - 1)^+ + E(1 - D_t)^+, \\ &\qquad 10\ E(D_t - 2)^+ + E(2 - D_t)^+, \\ &\qquad\qquad 10\ E(D_t - 3)^+ + E(3 - D_t)^+] \end{aligned}$$

(Notice that our states are labeled 1,2,3 in this problem.)

Complex Cost Functions: Example

EXAMPLE 3, continued

To compute the expected cost vector

$$E[C] = [10\ E(D_t - 1)^+ + E(1 - D_t)^+,\ 10\ E(D_t - 2)^+ + E(2 - D_t)^+,\ 10\ E(D_t - 3)^+ + E(3 - D_t)^+]$$

we first need to compute the following expected value for i =1,2,3:

$$E[(D_t - i)^+] = \sum_{k=0}^{\infty} p_k (k - i)^+ = \sum_{k=i+1}^{\infty} p_k (k - i)$$

In doing so, we get

$$E[(D_t - 1)^+] = \sum_{k=2}^{\infty} p_k (k - 1) = .2(2 - 1) = 0.2$$

$$E[(D_t - 2)^+] = \sum_{k=3}^{\infty} p_k (k - 2) = 0$$

$$E[(D_t - 3)^+] = \sum_{k=4}^{\infty} p_k (k - 3) = 0$$

Complex Cost Functions: Example

EXAMPLE 3, continued

We also need to compute the following expected value for i=1,2,3:

$$E[(i - D_t)^+] = \sum_{k=0}^{\infty} p_k (i - k)^+ = \sum_{k=0}^{i-1} p_k (i - k)$$

This yields:

$$E[(1 - D_t)^+] = \sum_{k=0}^{\infty} p_k (1 - k)^+ = \sum_{k=0}^{0} p_k (1 - k) = p_0 = 0.2$$

$$E[(2 - D_t)^+] = \sum_{k=0}^{\infty} p_k (2 - k)^+ = \sum_{k=0}^{1} p_k (2 - k) = 2p_0 + p_1 = 1$$

$$E[(3 - D_t)^+] = \sum_{k=0}^{\infty} p_k (3 - k)^+ = \sum_{k=0}^{2} p_k (3 - k) = 3p_0 + 2p_1 + p_2 = 2$$

Now, we can write

$$E[C] = [10 \cdot (0.2) + 0.2,\ 10 \cdot (0) + 1,\ 10 \cdot (0) + 2] = [2.2, 1, 2]$$

Using the steady state distribution $\pi_1 = 13/41$, $\pi_2 = 12/41$, $\pi_3 = 16/41$
we can now compute the long run expected average cost per period:

$$\lim_{n \to \infty} E\left[\frac{1}{n} \sum_{t=0}^{n-1} C(X_t, D_t)\right] = \sum_{j=0}^{S} \pi_j E[C(j, Y_t)] = \pi E[C] = 1.76$$

Finite Absorbing Chains

Our results about long run probabilities, long run average probabilities, and long run expected average cost per period hold for positive recurrent, irreducible chains.

However, not all interesting Markov Chains fall into this category. Sometimes interesting applications have some absorbing states and some transient states. Such chains are not irreducible. But we might want to answer questions like:

- How long does it take until the process gets “absorbed”?
- Which of the absorbing states is most likely?

We are now going to consider chains with finite state space in which all states are either absorbing or transient. We’ll call these finite absorbing chains.

Finite Absorbing Chains: Example

EXAMPLE 8: Accounts Receivable (Hillier & Lieberman)

A department store expects customers to pay their bills within 30 days. It classifies the balance on a customer’s bill as fully paid, 1 month late, 2 months late, or bad debt. Accounts are monitored monthly. A customer who is late but makes a partial payment this month is viewed as being 1 month late next month. Customers more than 2 months late have their bills sent to a collection agency. Customers who are fully paid and then make new purchases are viewed as “new” customers.

Define the states as 0 (fully paid), 1 (1 month late), 2 (2 months late), and 3 (bad debt). Transition probabilities are as follows:

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ .7 & .2 & .1 & 0 \\ .5 & .1 & 0 & .4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

note: slightly different from example in book!

The department store is interested the probability that a customer will end up as bad debt, given that his account is currently 1 month late.

Finite Absorbing Chains: Transition Matrix Structure

A finite absorbing chain has a finite set of *a* absorbing states and a finite set of *t* transient states. By rearranging the ordering of states, its transition matrix can be written as:

$$
\boldsymbol{P} = \begin{array}{r c c}
 & \textbf{\textit{a} absorbing columns} & \textbf{\textit{t} transient columns} \\
\textbf{\textit{a} absorbing rows} & \left[\begin{array}{c} I \\ R \end{array}\right. & \left.\begin{array}{c} 0 \\ T \end{array}\right] \\
\textbf{\textit{t} transient rows} & &
\end{array}
$$

Transition Matrix Structure: Example

EXAMPLE 8, continued

Reorder the states to be

0: fully paid

1: bad debt

2: 1 month late

3: 2 months late

$$
\begin{array}{l}
\text{fully paid} \\
\text{bad debt} \\
\text{1 month late} \\
\text{2 months late}
\end{array}
\begin{bmatrix}
\boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}^{\,I} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\
\boxed{\begin{matrix} .7 & 0 \\ .5 & .4 \end{matrix}}_{\,R} & \boxed{\begin{matrix} .2 & .1 \\ .1 & 0 \end{matrix}}_{\,T}
\end{bmatrix}
$$

We label the elements of the matrices *R* and *T* consistently with the elements of *P.*

$$R = \begin{bmatrix} R_{20} & R_{21} \\ R_{30} & R_{31} \end{bmatrix} = \begin{bmatrix} .7 & 0 \\ .5 & .4 \end{bmatrix}$$

$$T = \begin{bmatrix} T_{22} & T_{23} \\ T_{32} & T_{33} \end{bmatrix} = \begin{bmatrix} .2 & .1 \\ .1 & 0 \end{bmatrix}$$

Absorption Probabilities

Recall the first passage probability from i to j, denoted *by* $f_{ij}^{(n)}$. This is the probability that, starting in i, the first transition into state j occurs at time *n.*

$$f_{ij}^{(0)} = \begin{cases} 0 \text{ if } i \neq j \\ 1 \text{ if } i = j \end{cases} \qquad \begin{aligned} f_{ij}^{(n)} &= P\{T_{ij} = n\} \\ &= P\{X_n = j, X_{n-1} \neq j, \ldots, X_1 \neq j \mid X_0 = i\} \end{aligned}$$

Also recall that f_{ij} is the probability that the process will *eventually* visit state j given that it starts in state i.

$$f_{ij} = \sum_{n=1}^{\infty} f_{ij}^{(n)}$$

In a finite absorbing chain, when j is an absorbing state, then f_{ij} has a new interpretation: it is the probability that the process is eventually absorbed into state j, given that it started in state $i.$ If the starting state i is also absorbing, then clearly

$$f_{ii} = 1 \quad \text{and} \quad f_{ij} = 0 \text{ if } i \neq j$$

What's really interesting is when i is transient and j is absorbing.

Computing Absorption Probabilities

If i is a transient state and j is an absorbing state, then to compute the probability f_{ij} that the process will *eventually* be absorbed in state j given that it starts in state i, we use our standard trick of conditioning on the first step of the process.

$$\begin{aligned} f_{ij} &= \sum_{k=0}^{S} p_{ik} f_{kj} \\ &= p_{ij} f_{jj} + \sum_{\substack{k=0 \\ k \neq j}}^{S} p_{ik} f_{kj} = p_{ij} + \sum_{\substack{k=0 \\ k \neq j}}^{S} p_{ik} f_{kj} \end{aligned}$$

Notice that for all states k that are absorbing and different from k, $f_{kj} = 0$. That means that the only nonzero terms in the sum correspond to transient values of k. So we can rewrite f_{ij} again as

$$f_{ij} = p_{ij} + \sum_{k \text{ transient}} p_{ik} f_{kj}$$

for all pairs i,j where i is transient and j is absorbing.

Computing Absorption Probabilities: Example

EXAMPLE 8, continued

What's the probability that a customer whose payment is 1 month late will eventually become a bad debt customer? The answer is f_{21}.

We'll use the equations $f_{ij} = p_{ij} + \sum_{k \text{ transient}} p_{ik} f_{kj}$ to find it.

$$f_{21} = p_{21} + \sum_{k \text{ transient}} p_{2k} f_{k1} = p_{21} + p_{22} f_{21} + p_{23} f_{31} = 0 + (.2) f_{21} + (.1) f_{31}$$

$$f_{31} = p_{31} + \sum_{k \text{ transient}} p_{3k} f_{k1} = p_{31} + p_{32} f_{21} + p_{33} f_{31} = .4 + (.1) f_{21} + (0) f_{31}$$

Simplifying gives: $f_{21} = (.125) f_{31}$ $\qquad f_{31} = .4 + (.1) f_{21}$

Which results in: $f_{31} = .405$ $\qquad f_{21} = .051$

There is about a 5.1% chance that a customer currently 1 month late will become a bad debt customer.

Computing Absorption Probabilities using Matrix Algebra

Let's look closer at this equation while recalling the transition matrix of a finite absorbing chain. $\quad f_{ij} = p_{ij} + \sum_{k \text{ transient}} p_{ik} f_{kj}$

$$\boldsymbol{P} = \begin{array}{r} \textbf{\textit{a} absorbing rows} \\ \textbf{\textit{t} transient rows} \\ \\ \end{array} \begin{array}{c} \begin{bmatrix} I & 0 \\ R & T \end{bmatrix} \\ \begin{array}{cc} \textbf{\textit{a} absorbing columns} & \textbf{\textit{t} transient columns} \end{array} \end{array}$$

Let F represent the (t x a) matrix of absorption probabilities (f_{ij}) from transient states i to absorbing states j.

Then our equation can be rewritten in matrix form as

$$F = R + TF \qquad \text{or}$$

$$F = (I - T)^{-1} R$$

Computing Absorption Probabilities using Matrix Algebra: Example

EXAMPLE 8, continued

Let's use matrix algebra to recompute the probability that a customer whose payment is 1 month late will eventually become a bad debt customer.

$$\begin{array}{l} \text{fully paid} \\ \text{bad debt} \\ \text{1 month late} \\ \text{2 months late} \end{array} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ .7 & 0 & .2 & .1 \\ .5 & .4 & .1 & 0 \end{bmatrix} \qquad \begin{bmatrix} \boxed{I} & 0 \\ \boxed{R} & \boxed{T} \end{bmatrix}$$

We first compute $I - T$: $\quad I - T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} .2 & .1 \\ .1 & 0 \end{bmatrix} = \begin{bmatrix} .8 & -.1 \\ -.1 & 1 \end{bmatrix}$

Then, we compute $(I - T)^{-1}$: $\quad (I - T)^{-1} = \begin{bmatrix} 1.2658 & 0.1266 \\ 0.1266 & 1.0127 \end{bmatrix}$

$$F = \begin{bmatrix} f_{20} & f_{21} \\ f_{30} & f_{31} \end{bmatrix} = (I - T)^{-1} R = \begin{bmatrix} 1.2658 & 0.1266 \\ 0.1266 & 1.0127 \end{bmatrix} \begin{bmatrix} .7 & 0 \\ .5 & .4 \end{bmatrix} = \begin{bmatrix} .949 & .051 \\ .595 & .405 \end{bmatrix}$$

Limiting Distributions of Finite Absorbing Chains

In a finite absorbing chain, what happens to the matrix $P^{(n)}$ as n gets large?

$$\boldsymbol{P} = \begin{array}{r} \textbf{\textit{a} absorbing rows} \\ \textbf{\textit{t} transient rows} \end{array} \begin{bmatrix} I & 0 \\ R & T \end{bmatrix}$$

$$\begin{array}{cc} \textbf{\textit{a} absorbing columns} & \textbf{\textit{t} transient columns} \end{array}$$

$$P^2 = \begin{bmatrix} I & 0 \\ R & T \end{bmatrix} \begin{bmatrix} I & 0 \\ R & T \end{bmatrix} = \begin{bmatrix} I & 0 \\ R + TR & T^2 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} I & 0 \\ R & T \end{bmatrix} \begin{bmatrix} I & 0 \\ R + TR & T^2 \end{bmatrix} = \begin{bmatrix} I & 0 \\ (I + T + T^2)R & T^3 \end{bmatrix}$$

$$P^n = \begin{bmatrix} I & 0 \\ (I + T + T^2 + \ldots + T^n)R & T^n \end{bmatrix}$$

It can be shown that for a matrix like T, in which all row sums are at most 1, and at least one row has sum less than 1, the following things are true:

$$\lim_{n \to \infty} \sum_{i=0}^{n} T^i = (I - T)^{-1} \qquad\qquad \lim_{n \to \infty} T^n = 0$$

Limiting Distributions of Finite Absorbing Chains

As a result, as n tends towards infinity,

$$P^n = \begin{bmatrix} I & 0 \\ (I + T + T^2 + \dots + T^n)R & T^n \end{bmatrix} \to \begin{array}{cc|l} \textbf{\textit{a} absorbing columns} & \textbf{\textit{t} transient columns} & \\ \hline \begin{bmatrix} I \\ \underbrace{(I - T)^{-1}R}_{= F} \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \begin{array}{l} \textbf{\textit{a} absorbing rows} \\ \textbf{\textit{t} transient rows} \end{array} \end{array}$$

Notice the following things:

- Unlike in the case of irreducible positive recurrent chains, the rows of P^n are not identical. Thus, the limiting probabilities are dependent on the initial conditions.
- If the chain starts in an absorbing state, it will stay there forever
- After an infinite number of steps, the probability of visiting a transient state is 0
- Starting from a transient state i, the limiting probability of visiting an absorbing state j is the absorption probability f_{ij} from submatrix $F = (I - T)^{-1} R$.

Limiting Distributions of Finite Absorbing Chains: Example

EXAMPLE 8, continued: Loss expectancy rates

Suppose the department store is currently owed $20 million by customers who are 1 month late, and $10 million by customers who are 2 months late. How much do they expect to lose of the money in these accounts?

For a randomly selected dollar currently owed, there is a probability of 20/30= 2/3 that it is 1 month late, and 10/30=1/3 that it is 2 months late. Thus, the initial distribution of the system over the transient states is

$$(Q_2, Q_3) = (20/30, 10/30) = (.667, .333)$$

For a random dollar, the probability that it will become bad debt eventually is

$$\lim_{n \to \infty} P(X_n = 1) = \lim_{n \to \infty} \sum_{i=0}^{s} Q_i p_{i1}^{(n)} = \lim_{n \to \infty} \sum_{\text{transient } i} Q_i p_{i1}^{(n)} = \sum_{\text{transient } i} Q_i f_{i1}$$

$$= Q_2 f_{21} + Q_3 f_{31} = (.667)(0.051) + (.333)(.405) = .169$$

Thus, out of the $30 million initially owed, the expected loss is ($30 million) (0.169) = $5.07 million.

Expected number of visits to a transient state

Define m_{ij} to be the expected number of visits to a transient state *j* given that you started in a transient state *i.* It turns out that m_{ij} is the *ij*th element of the matrix $(I-T)^{-1}$. (with the matrix rows and columns labeled consistently with the elements of *P*.)

EXAMPLE 8, continued

For how many months is a customer who is currently 1 month late expected to be considered 1 month late? Answer: m_{22} =1.258 months.

$$(I-T)^{-1} = \begin{array}{ll} \begin{bmatrix} 1.2658 & 0.1266 \\ 0.1266 & 1.0127 \end{bmatrix} & \begin{array}{l} \text{1 month late (state 2)} \\ \text{2 months late (state 3)} \end{array} \end{array}$$