Assignment 7 Solution

1. We have an M/G/1 Queue having exponential arrival rate $\lambda = 0.5$ per minute and i.i.d. service times with mean $1/\mu$ and variance σ^2. Recall that for $\rho = \lambda/\mu$, we know by the Pollaczek-Khintchine formula that

$$W_q = \frac{\lambda^2\sigma^2 + \rho^2}{2\lambda(1-\rho)}, \quad L = \lambda W = \frac{\lambda^2\sigma^2 + \rho^2}{2(1-\rho)} + \rho$$

For Bob, we have $1/\mu = 1.5$, $\sigma^2 = 3^2/12 = 0.75$ and $\rho = 0.75$. This yields

$$W_q = \frac{0.5^2 \cdot 0.75 + 0.75^2}{2 \cdot 0.75 \cdot (1 - 0.75)} = 3, \quad L = \frac{0.5^2 \cdot 0.75 + 0.75^2}{2 \cdot (1 - 0.75)} + 0.75 = 2.25$$

When Bob is working, the expected waiting time of a customer in the queue is 3 minutes, and the expected number of cars in the drive-thru is 2.25.

For Charlie, we have $1/\mu = 1.55$, $\sigma^2 = 0$ and $\rho = 1.55 \cdot 0.5 = 0.775$. This yields

$$W_q = \frac{0.5^2 \cdot 0 + 0.775^2}{2 \cdot 0.75 \cdot (1 - 0.775)} = 2.67, \quad L = \frac{0.5^2 \cdot 0 + 0.775^2}{2 \cdot (1 - 0.775)} + 0.775 = 2.11$$

When Charlie is working, the expected waiting time of a customer in the queue is 2.67 minutes, and the expected number of cars in the drive-thru is 2.11.

Although Bob works faster on average, Charlies consistency leads to shorter lines and shorter waits. Therefore, as a manager, you should choose Charlie to work the window.

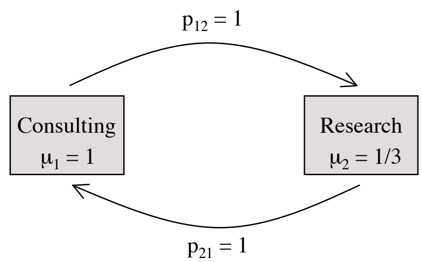
2. (a) This is a M/G/1 queue, since we are given no additional information on the distribution of the service time. The expected interarrival time is $1/\lambda = 6$ minutes, and the expected service time is $1/\mu = 5$ minutes, then, the traffic intensity is $\rho = 5/6$ The percentage of time the operator is busy is

$$1 - \pi_0 = 1 - (1 - \rho) = \rho = \frac{5}{6}$$

(b) The likelihood that a customer needs to wait for more than 10 minutes is $P(\mathcal{W}_q > 10)$, but since we cannot compute this quantity directly, and $\rho = 5/6$ is close to one, we can use a heavy-traffic approximation. Note that since the arrivals occur according to a Poisson Process, the interarrival times are Exponential (λ) r.v., so $\sigma_A^2 = 1/\lambda^2$ and $m_A = 1/\lambda$. Then,

$$P(\mathcal{W}_q > 10) \approx P\left(\frac{(\sigma_A^2 + \sigma_S^2)}{2(m_A - m_S)}\, exp(1) > 10\right) = P\left(\frac{(36+9)}{2(6-5)}\, exp(1) > 10\right)$$
$$= P\left(exp(1) > \frac{4}{9}\right) = e^{-4/9} = 0.6412$$

(a) The customers are the two members of the team, and the stations are the projects they can work on. Let station 1 be the "Consulting" project, and let station 2 be the "Research" project. Station 1 is a M/M/1 queue with service rate $\mu = 1$, and station 2 is either a $M/M/\infty$ queue or a M/M/2 queue (since there are only two people in the system) with service rate $\mu = 1/3$. The corresponding diagram is the following:



(b) We want to compute the steady-state probability of having one research and one consulting project, that is, $\pi_{1.1}$. The traffic equations for this network are

$$\begin{aligned}\gamma_1 &= p_{21}\gamma_2 = \gamma_2\\ \gamma_2 &= p_{12}\gamma_1 = \gamma_1\\ 1 &= \gamma_1 + \gamma_2\end{aligned}$$

which has solution $\gamma_1 = \gamma_2 = 1/2$.

We now need to compute the steady-state distribution for each station in isolation. For station 1:

$$\pi_k^1 = (1 - \gamma_1/\mu_1)\left(\frac{\gamma_1}{\mu_1}\right)^k, \qquad \text{for } k = 0, 1, \ldots,$$

and for station 2:

$$\pi_k^2 = e^{-\gamma_2/\mu_2}\frac{(\gamma_2/\mu_2)^k}{k!}, \text{ for } k = 0, 1, \ldots, \qquad \text{(if we consider a M/M/}\infty\text{)}$$

$$\pi_k^2 = \begin{cases} \left(\sum_{i=0}^{1}\frac{(\gamma_2/\mu_2)^i}{i!} + \frac{(\gamma_2/\mu_2)^2}{2!}\frac{1}{1-\gamma_2/(2\mu_2)}\right)^{-1}, & k = 0\\ \frac{(\gamma_2/\mu_2)^k}{k!}\pi_0^2, & k = 1, 2\\ \frac{(\gamma_2/\mu_2)^k}{2!2^{k-2}}\pi_0^2, & k = 3, 4, \ldots \end{cases} \qquad \text{(if we consider a M/M/2)}$$

The above simplifies to

$$\pi_k^1 = \left(\frac{1}{2}\right)^{k+1}, \qquad \text{for } k = 0, 1, \ldots,$$

$$\pi_k^2 = e^{-3/2}\frac{(3/2)^k}{k!}, \text{ for } k = 0, 1, \ldots, \qquad \text{(if we consider a M/M/}\infty\text{)}$$

$$\pi_k^2 = \begin{cases} \left(\frac{3}{2}\right)^k \frac{1}{7k!}, & k = 0, 1, 2\\ \left(\frac{3}{4}\right)^k \frac{2}{7}, & k = 3, 4, \ldots \end{cases} \qquad \text{(if we consider a M/M/2)}$$

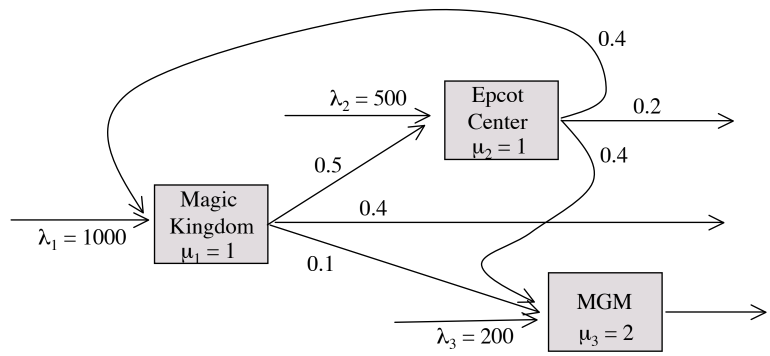
Note that regardless of what model for station 2 we use,

$$\pi_{1,1} = \frac{\pi_1^1 \pi_1^2}{\pi_0^1 \pi_2^2 + \pi_1^1 \pi_1^2 + \pi_2^1 \pi_0^2}$$

It can be verified that for both models (M/M/2 or M/M/∞ for station 2), the above is equal to

$$\pi_{1,1} = \frac{6}{17}.$$

3. (a) The diagram is the following



(b) Call Magic Kingdom station 1, Epcot Center station 2, and MGM station 3, and note that each station is a M/M/∞ queue. The arrival rates for each station satisfy the following system of equations:

$$\begin{aligned}\gamma_1 &= 1000 + 0.4\gamma_2 \\ \gamma_2 &= 500 + 0.5\gamma_1 \\ \gamma_3 &= 200 + 0.1\gamma_1 + 0.4\gamma_2\end{aligned}$$

which has solution

$$(\gamma_1, \gamma_2, \gamma_3) = (1500, 1250, 850)$$

(c) The average rate of departure from MGM Studios is $\lambda_3 = 850$ per day.

(d) The average daily income is $40L$, where $L = L_1 + L_2 + L_3$, and L_i is the steady-state average number of people in station i. For station i the steady-state probabilities are given by

$$\pi_k^i = \frac{(\gamma_i/\mu_i)^k}{k!} e^{-\gamma_i/\mu_i}$$

so the steady-state average number of customers is

$$L_i = \sum_{k=0}^{\infty} k\pi_k^i = \sum_{k=1}^{\infty} k\frac{(\gamma_i/\mu_i)^k}{k!} e^{-\lambda_i/\mu_i} = \frac{\gamma_i}{\mu_i} \sum_{k=1}^{\infty} \frac{(\gamma_i/\mu_i)^{k-1}}{(k-1)!} e^{-\gamma_i/\mu_i} = \frac{\gamma_i}{\mu_i}$$

It follows that

$$40L = 40\left(\frac{\gamma_1}{\mu_1} + \frac{\gamma_2}{\mu_2} + \frac{\gamma_3}{\mu_3}\right) = 40(3175) = 127,000$$

(e) The average daily income would then be

$$40L_1 + 35L_2 + 50L_3 = 125,000$$

so it would be less profitable.

4. Let X_t be the number of customers in-system at time t. The following table contains all the information that we need to keep track of the number of customers in the system at any given time:

t	Arrival	Departure	X_t	Service time left of customer being served	Customers in the system
90	✓		1	20	**1** |
100	✓		2	10	**1** | 2
110		✓	1	30	**2** |
120	✓		2	20	**2** | 3
130	✓		3	10	**2** | 3 4
140		✓	2	40	**4** | 3
170	✓		3	10	**4** | 3 5
180		✓	2	10	**5** | 3
190		✓	1	30	**3** |
200	✓		2	20	**3** | 6
210	✓		3	10	**3** | 6 7
220		✓	2	30	**7** | 6
250		✓	1	50	**6** |
260	✓		2	40	**6** | 8

At time $t = 240$ minutes (4 hrs), there were 2 customers in the system.