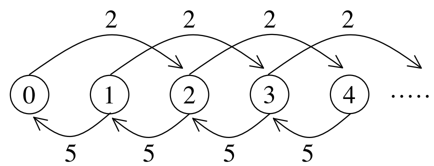
Assignment 6 Solution

1. (a) The rate diagram is:



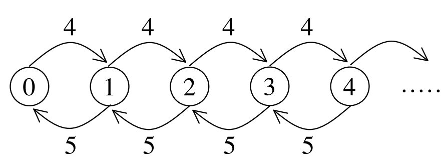
(b) The rate matrix is

$$Q = \begin{bmatrix} -2 & 0 & 2 & 0 & 0 & 0 & \dots \\ 5 & -7 & 0 & 2 & 0 & 0 & \dots \\ 0 & 5 & -7 & 0 & 2 & 0 & \dots \\ 0 & 0 & 5 & -7 & 0 & 2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

The balance equations are

$$\begin{aligned} 2\pi_0 &= 5\pi_1 \\ 7\pi_1 &= 5\pi_2 \\ 7\pi_k &= 2\pi_{k-2} + 5\pi_{k+1} \quad \text{for } k = 2, 3, \dots \end{aligned}$$

(c) The rate diagram for the birth-and-death process is:



(d) There are several ways of answering this question, and we give two of them here.

i. *Simulation:* We simulate the original Markov chain (the one having infinite state space) up to time t a large number of times, say N. Let $X_t^{(i)}$, $i = 1, \dots, N$, be the number-in-system at time t observed on the i^{th} simulation. An unbiased estimator for the expected number-in-system at time t, given that the Markov chain started in state x at time zero, is then $E_x X_t = \frac{1}{N} \sum_{i=1}^{N} X_t^{(i)}$; confidence intervals can be computed in the usual way. A Matlab code to estimate $E_x X_t$, with its corresponding confidence intervals, can be found in the course's webpage. Below we give 95% confidence intervals for the expected number-in-system at time $t = 1, 10, 100$, given that the chain started empty at time zero:

$$E_0 X_1 \in (2.3132, 2.6668), \qquad E_0 X_{10} \in (4.7696, 5.5864), \qquad E_0 X_{100} \in (5.2156, 6.2724)$$

ii. *Truncation:* We start by truncating the state space, say to a maximum capacity of 10 customers (as suggested in the hint we gave). The rate matrix for the truncated queue is

$$\hat{Q} = \begin{bmatrix} -2 & 0 & 2 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 5 & -7 & 0 & 2 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 5 & -7 & 0 & 2 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 5 & -7 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 5 & -7 & 2 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 5 & -5 \end{bmatrix}$$

We know that the transient distribution $P(t) = (P(t;i,j))$ is given by

$$P(t) = \sum_{k=0}^{\infty} \frac{\hat{Q}^k t^k}{k!}$$

If Q was diagonalizable, we would be able to compute analytically the above series, but unfortunately, that is not the case, so we have to do it numerically (using Matlab). Once $P(t)$ has been computed, the expected number-in-system at time t, given that the Markov chain started in state x at time zero, is given by

$$E_x X_t = \sum_{y=0}^{10} y P(t; x, y)$$

where $P(t;i,j)$ is the element of $P(t)$ corresponding to the i^{th} row and j^{th} column. Below we give the results for $t = 1, 10, 100$, given that the chain started empty at time zero:

$$E_0 X_1 = 1.8155, \qquad E_0 X_{10} = 3.3521, \qquad E_0 X_{100} = 3.3612$$

The code used to obtain these can be downloaded from the course's website.

(e) The transition matrix for the reduced model is the one given in part (d)ii. The steady-state distribution of the chain can by obtained by solving the system of linear equations:

$$\pi \hat{Q} = 0$$
$$\sum_{n=0}^{10} \pi_n = 1$$

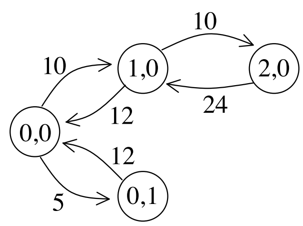
where $\pi = (\pi_0, \pi_1, \dots, \pi_{10})$. The solution to this system is:

$$\pi = (0.2466, 0.0987, 0.1381, 0.0947, 0.0931, 0.0751, 0.0673, 0.0570, 0.04971, 0.0427, 0.0369)$$

And the steady-state expected number in system is given by:

$$E[X_\infty] = \sum_{y=0}^{10} y\pi_y = 3.3612$$

2. (a) Let $X_t = (Y_t, Z_t)$, where Y_t is the number of customers of class 1 at time t, and Z_t is the number of customers of class 2 at time t. The state space for this continuous time Markov chain is $\{(0,0),(1,0),(2,0),(0,1)\}$ (recall that customers of class 2 require the simultaneous use of both servers). The rate diagram for this model is (all the rates are measured in customers per hour):



(b) From the rate diagram we can see that this can be fitted into a birth-and-death process by just renaming the states as $\{0,1,2,3\} = \{(0,1),(0,0),(1,0),(2,0)\}$, and noticing that transitions only occur between adjacent states.

(c) From the birth-and-death results we know that the steady-state probabilities for the renamed states are:

$$\pi_0 = 0.1604, \quad \pi_1 = 0.3850, \quad \pi_2 = 0.3209, \quad \pi_3 = 0.1337$$

which translate in the original notation to

$$\pi_{(0,0)} = 0.3850, \quad \pi_{(1,0)} = 0.3209, \quad \pi_{(2,0)} = 0.1337, \quad \pi_{(0,1)} = 0.1604$$

The above are the steady-state joint distributions for (Y_t, Z_t).

(d) Customers of class 1 are lost when the chain is in states $(2,0)$ or $(0,1)$; and customers of class 2 are lost when the chain is in states $(1,0)$, $(2,0)$ or $(0,1)$. Therefore, the expected fraction of arrivals who are unable to enter the system is:

$$\begin{aligned} \pi_{(2,0)} + \pi_{(0,1)} &= 0.2941 \qquad \text{for class 1} \\ 1 - \pi_{(0,0)} &= 0.6150 \qquad \text{for class 2} \end{aligned}$$

3. We first define our CTMC to be X_t = number of broken machines at time t, and note that our possible states are $\{0,1,2\}$. The long-run proportion that there is a working machine is $P_0 + P_1$, where the P_i's are the limiting probabilities. To compute them we solve the system

$$\begin{aligned} v_j P_j &= \sum_{k \neq j} q_{kj} P_k, \qquad \text{for all states } j \\ \sum_j P_j &= 1 \end{aligned}$$

where

$$\begin{aligned} v_0 &= \lambda \\ v_1 &= \lambda + \mu \\ v_2 &= \mu \\ Q &= (q_{ij}) = \begin{bmatrix} 0 & \lambda & 0 \\ \mu & 0 & \lambda \\ 0 & \mu & 0 \end{bmatrix} \end{aligned}$$

which has solution

$$(P_0, P_1, P_2) = \left(\frac{\mu^2}{\lambda^2 + \lambda\mu + \mu^2}, \frac{\lambda\mu}{\lambda^2 + \lambda\mu + \mu^2}, \frac{\lambda^2}{\lambda^2 + \lambda\mu + \mu^2} \right)$$

The proportion of time there's a working machine is then

$$P_0 + P_1 = \frac{\mu(\lambda + \mu)}{\lambda^2 + \lambda\mu + \mu^2}$$

4. We first consider the chain X_t = number of customers in the system at time t, and we note that when there's one customer in the system we need to distinguish which server he is in. An appropriate state space for this chain is $\mathcal{S} = \{0, (1,0), (0,1), 2, 3, \dots\}$, where (1, 0) corresponds to having only one customer in server 1 and (0, 1) corresponds to having only one customer in server 2. We first identify the quantities of interest (let $\gamma = \mu_1 + \mu_2$):

$$\begin{aligned}
Q &= (q_{ij}) = \begin{bmatrix} 0 & \lambda/2 & \lambda/2 & 0 & 0 & 0 & 0 & 0 & \dots \\ \mu_1 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & \dots \\ \mu_2 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & \dots \\ 0 & \mu_2 & \mu_1 & 0 & \lambda & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & \gamma & 0 & \lambda & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \gamma & 0 & \lambda & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & \gamma & 0 & \lambda & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix} \\
v_0 &= \lambda \\
v_{(1,0)} &= \lambda + \mu_1 \\
v_{(0,1)} &= \lambda + \mu_2 \\
v_i &= \lambda + \gamma \qquad \text{for } i = 2, 3, 4, \dots
\end{aligned}$$

To compute the limiting probabilities P_i we solve the system of equations

$$\begin{aligned}
v_j P_j &= \sum_{k \neq j} q_{kj} P_k, \qquad \text{for all states } j \\
\sum_j P_j &= 1
\end{aligned}$$

Substituting the values of the the v_i's and the q_{ij}'s gives

$$\begin{aligned}
\lambda P_0 &= \mu_1 P_{(1,0)} + \mu_2 P_{(0,1)} \\
(\lambda + \mu_1) P_{(1,0)} &= \frac{\lambda}{2} P_0 + \mu_2 P_2 \\
(\lambda + \mu_2) P_{(0,1)} &= \frac{\lambda}{2} P_0 + \mu_1 P_2 \\
(\lambda + \gamma) P_2 &= \lambda (P_{(0,1)} + P_{(1,0)}) + \gamma P_3 \\
&\vdots \\
(\lambda + \gamma) P_k &= \lambda P_{k-1} + \gamma P_{k+1} \qquad \text{for } k = 3, 4, \ldots \\
&\vdots \\
\sum_j P_j &= 1
\end{aligned}$$

We know that the "general" equation for states $k = 3, 4, \ldots$ has a solution of the form

$$P_k = C \left(\frac{\lambda}{\gamma} \right)^k,$$

where C is a constant. To find what this constant should be we need to solve the first 4 equations. First we note that the fourth equation has almost the general form, so we deduce that

$$P_2 = C \left(\frac{\lambda}{\gamma} \right)^2 \qquad \text{and} \qquad P_{(0,1)} + P_{(1,0)} = C \left(\frac{\lambda}{\gamma} \right).$$

Now by substituting $P_{(0,1)} = C \left(\frac{\lambda}{\gamma} \right) - P_{(1,0)}$ into the first equation gives

$$P_{(1,0)} = \frac{\lambda}{\mu_1 - \mu_2} P_0 - \frac{C \lambda \mu_2}{\gamma (\mu_1 - \mu_2)}$$

and substituting this in the second equation gives

$$P_0 = \frac{C 2 \mu_1 \mu_2}{\gamma^2}$$

Finally,

$$P_{(1,0)} = \frac{C \lambda \mu_2}{\gamma^2}, \qquad P_{(0,1)} = \frac{C \lambda \mu_1}{\gamma^2},$$

and C is determined by the equation

$$P_0 + P_{(1,0)} + P_{(0,1)} + P_2 + P_3 + \cdots = 1,$$

which has as solution

$$C = \left(\frac{2 \mu_1 \mu_2}{\gamma^2} - 1 + \frac{\gamma}{\gamma - \lambda} \right)^{-1}$$