# Assignment 5 Solution

1. Let $X_i$ be the lifetime of component $i$. Then $X_i$ has exponential distribution with parameter $\lambda_i$ and the lifetime of the machine $M$ is given by $M = \min_{1\le i\le n} X_i$. It follows that for any $x \ge 0$,

$$P(M \ge x) = P(X_1 \ge x, \dots, X_n \ge x) = \prod_{i=1}^{n} P(X_i \ge x) = \prod_{i=1}^{n} e^{-\lambda_i x} = e^{-(\lambda_1+\cdots+\lambda_n)x}$$

The second step follows from the independence of $X_i$'s. Therefore, $M$ has exponential distribution with parameter

$$\lambda = \lambda_1 + \cdots + \lambda_n.$$

2. (a) Since the service times of all the customers is exactly 10 minutes, $A$ and $B$ finish at the same time (exactly 10 minutes after they came in), and $C$ starts his service when $A$ and $B$ have both left. Therefore, the probability that $A$ will still be there is 0.

   (b) Let $T_A$, $T_B$ and $T_C$ be the service times of $A$, $B$, and $C$, respectively. Each of the $T_i$s can take values 1, 2, or 3 with the same probability. Note that $A$ will still be in the post office after the other two have left if $T_A > T_B + T_C$, so

$$\begin{aligned} &P(T_A > T_B + T_C) \\ &= \sum_{i=1}^{3}\sum_{j=1}^{3} P(T_A > i + j | T_B = i, T_C = j)\,\frac{1}{9} \\ &= \frac{1}{9}[P(T_A > 2) + 2P(T_A > 3) + 3P(T_A > 4) + 2P(T_A > 5) + P(T_A > 6)] \\ &= \frac{1}{9}P(T_A > 2) = \frac{1}{9}P(T_A = 3) = \frac{1}{27} \end{aligned}$$

   (c) If the $T_i$'s are Exponential ($\mu$) random variables, we can use the memoryless property and say that if $T_A > T_B + T_C$, it has to be that $T_A > T_B$, which happens with probability 1/2. Since $T_A$ is memoryless, after $B$ has left, the additional time $A$ will be in service is still Exponential ($\mu$), so now it has to occur that $T_A > T_C$, which happens again with probability 1/2. Therefore, the probability that $A$ will still be in service after the other two have left is 1/4.

   A mathematical way of seeing this is the following:

$$\begin{aligned} P(T_A > T_B + T_C) &= \int_0^\infty \int_0^\infty P(T_A > x + y | T_B = x, T_C = y)\mu^2 e^{-\mu(x+y)} dx\, dy \\ &= \int_0^\infty \mu^2 e^{-\mu(x+y)} e^{-\mu(x+y)} dx\, dy = \int_0^\infty \frac{\mu}{2}\, e^{-2\mu y} \int_0^\infty 2\mu e^{-2\mu x} dx\, dy \\ &= \frac{1}{4}\int_0^\infty 2\mu e^{-\mu y} dy = \frac{1}{4} \end{aligned}$$

3. (a) The state space $E = \{1, 2, 3, 4\}$, where 1=working, 2=undergoing engine tune up, 3=undergoing a/c repair, and 4=undergoing braking system replacement.

(b) Using days as the time unit:

$$Q = \begin{bmatrix} -\frac{1}{20} & \frac{1}{20} & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -\frac{2}{3} & \frac{2}{3} \\ \frac{1}{2} & 0 & 0 & -\frac{1}{2} \end{bmatrix}$$

(c)

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Notice that the embedded DTMC is deterministic.

(d) Using days as the time unit:

$$\mathrm{E}[T_i] = \frac{1}{q_i} = -\frac{1}{q_{ii}} = \begin{cases} 20, & i = 1 \\ 1, & i = 2 \\ 3/2, & i = 3 \\ 2, & i = 4 \end{cases}$$

(e) No, the exponential distribution is not a realistic model for the lifetime of something that is subject to wear and tear.

(f) The chain is finite and irreducible, so it must be positive recurrent. Therefore a steady state distribution exists. Let $\pi = (\pi_1, \pi_2, \pi_3, \pi_4)$ be the steady-state distribution. Then we need to solve the following system of equations:

$$\pi Q = 0 \qquad \text{and} \qquad \sum_{i=1}^{4} \pi_i = 1$$

which gives us

$$\pi = (\frac{40}{49}, \frac{2}{49}, \frac{3}{49}, \frac{4}{49}) = (0.8163, 0.0408, 0.0612, 0.0816)$$

(g) Your car is the shop when the state is 2, 3 or 4, so the long run proportion of time that your car is in the shop is

$$\pi_2 + \pi_3 + \pi_4 = \frac{9}{49} = 0.184.$$

4. Let $X_t$ be the number of customers in the barbershop at time $t$, and note that $X_t$ takes only the values $\{0, 1, 2\}$. The arrival rate is $\lambda = 3$ and the service rate is $\mu = 4$. The rate matrix is given by

$$Q = \begin{bmatrix} -3 & 3 & 0 \\ 4 & -7 & 3 \\ 0 & 4 & -4 \end{bmatrix}.$$

(a) The transition probabilities $P(t)$ can be computed as

$$P(t) = e^{tQ} := \sum_{n=0}^{\infty} \frac{t^n Q^n}{n!}.$$

$P(1), P(5)$ and $P(10)$ can be calculated by using Matlab commands:

```
>> Q = [-3 3 0; 4 -7 3; 0 4 -4]

Q =

    -3     3     0
     4    -7     3
     0     4    -4

>> P1 = expm(1*Q)

P1 =

    0.4448    0.3227    0.2325
    0.4302    0.3246    0.2451
    0.4134    0.3269    0.2597

>> P5 = expm(5*Q)

P5 =

    0.4324    0.3243    0.2432
    0.4324    0.3243    0.2432
    0.4324    0.3243    0.2432

>> P10 = expm(10*Q)

P10 =

    0.4324    0.3243    0.2432
    0.4324    0.3243    0.2432
    0.4324    0.3243    0.2432
```

Now given that the Markov chain starts off at time 0 in state 0, the expected number of customers in the barbershop at times $t = 1$ hour, $t = 5$ hours and $t = 10$ hours are

$$\begin{aligned} E[X(1)|X(0)=0] &= \sum_{i=0}^{2} iP(1;1,i+1) = 0.7877 \\ E[X(5)|X(0)=0] &= \sum_{i=0}^{2} iP(5;1,i+1) = 0.8107 \\ E[X(10)|X(0)=0] &= \sum_{i=0}^{2} iP(10;1,i+1) = 0.8107 \end{aligned}$$

where $P(t;i,j)$ represents $(i,j)$th entry of $P(t)$.

(b) By solving the balance equations

$$\begin{aligned} 3\pi(0) &= 4\pi(1) \\ 7\pi(1) &= 3\pi(0) + 4\pi(2) \\ 4\pi(2) &= 3\pi(1) \end{aligned}$$

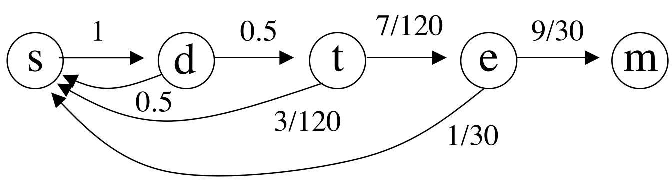
together with $\pi(0)+\pi(1)+\pi(2) = 1$, we can get $\pi(0) = 16/37$, $\pi(1) = 12/37$ and $\pi(2) = 9/37$. So, the average number of customers in the system is $0 \cdot \pi(0) + 1 \cdot \pi(1) + 2 \cdot \pi(2) = 30/37$.

(c) By PASTA, $\pi(0) + \pi(1) = 28/37$.

(d) By recomputing the answer from part (b) with $\mu = 8$, we get $\pi(0) = 64/97$, $\pi(1) = 24/97$ and $\pi(2) = 9/97$ and the proportion of customers that enter the shop is $\pi(0) + \pi(1) = 88/97$.

5. Let $X_t$ be Sam's relationship status at time $t$. The states for this continuous time Markov chain are {single (s), dating (d), steady (t), engaged (e), married (m)} (the "married" state is optional, depending on how you decide to model the chain.

(a) To compute the rates to and from the different states we use the holding times given in the problem: $\lambda = (\lambda(s), \lambda(d), \lambda(t), \lambda(e), \lambda(m)) = (1, 1, \frac{1}{12}, \frac{1}{3})$, and the transition probabilities. The rate diagram is:




(b) Limiting probabilities are defined to be

$$\lim_{t\to\infty} P(X_t = i), \qquad i = s, d, t, e, m,$$

so they are always well defined. In this case, since the "married" state is absorbent, we have that

$$\lim_{t\to\infty} P(X_t = i) = \begin{cases} 1, & \text{if } i = m \\ 0, & \text{otherwise.} \end{cases}$$

(c) The probability that Sam will marry Marcy is $(0.5)(0.7)(0.9) = 0.315$.