Assignment 5 - *Due Friday February 20th*

1. Consider a machine that consists of n components. The lifetime of component i has an exponential distribution with parameter λ_i for $i = 1, 2, \ldots, n$. The machine functions only while all of its components are functioning. Show that the lifetime of the machine is exponentially distributed with parameters
$$\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_n$$

2. Consider a post office with two clerks. Three people, A, B, and C, enter simultaneously. A and B go directly to the clerks, and C waits until either A or B leaves before he begins service. What is the probability that A is still in the post office after the other two have left when

   (a) the service time for each clerk is exactly (nonrandom) ten minutes?

   (b) the service times are i with probability $\frac{1}{3}$, $i = 1, 2, 3$?

   (c) the service times are exponential with mean $\frac{1}{\mu}$?

3. Your 1969 Ford Mustang requires frequent service. In fact, the time between required service are exponentially distributed with mean of 20 days. Each time it goes into the garage it needs a major overhaul consisting three operations that are carried out sequentially: engine tune-up, air conditioning and braking system replacement. The time to perform each operation follows an exponential distribution with mean times 1, 1.5 and 2 days, respectively. (Assume the times for the respective operations are mutually independent and there's no delay between operations.) Then the status of your car can be modeled as a CTMC.

   (a) What is the state space?

   (b) What is the transition rate matrix Q?

   (c) What is the transition probability matrix P for the embedded discrete-time Markov chain?

   (d) What is the expected holding time in state i, for all states $i \in E$.

   (e) Is it realistic to assume that the times between required service on a car are exponentially distributed? Explain.

   (f) Does a steady-state solution distribution exist for this CTMC? Explain. If so, compute it.

   (g) What's the expected proportion of time that your car is in the shop in the long run?

4. A small barbershop, operated by a single barber, has room for at most two customers. Potential customers arrive at a Poisson rate of three per hour, and the successive service times are independent exponential random variables with mean $\frac{1}{4}$ hour.

   (a) Compute the expected number of customers in the barbershop at times t=1 hour, 5 hours, 10 hours.

   (b) What is the average number of customers in the shop?

(c) What is the proportion of potential customers that enter the shop?

(d) If the barber could work twice as fast, how much more business would he do?

5. Sam, an MS&E major at Stanford, thinks of his love life as a continuous time Markov chain. Relationships, according to him, go through three different phases: "dating", "steady", and "engaged". Whenever Sam is not in a relationship, he is "single"; and if a relationship survives engagement, he will get "married" and stop dating. He models the amount of time he spends on each state as an Exponential random variable. Sam stays single for a period having mean 1 month. When he starts a new relationship, the dating phase also lasts for a period having mean 1 month. If Sam is going steady with someone, this will last for a period having mean 1 year. Finally, if Sam is engaged, the engagement will last for a period with mean 3 months.

   Any of Sam's relationships can only progress forward (i.e., from "dating" to "steady" to "engaged" to "married") or terminate, leaving Sam single. The probabilities of a relationship terminating from a particular state are:

   - dating: 0.50
   - steady: 0.30
   - engaged: 0.10

   Sam never gets into more than one relationship at a time.

   (a) Draw the rate diagram for Sam's dating process.

   (b) Does this Markov chain have limiting probabilities? Why or why not?

   (c) Sam just started dating Marcy today. What is the probability that he will marry Marcy?