Assignment 4 Solution

1. (a) The state space for $\{X_n\}$ is $\mathcal{S} = \{0, 1, 2, 3, 4\}$, and the transition matrix is

$$P = \begin{bmatrix} 0 \; .5 \; 0 \; 0 \; .5 \\ .5 \; 0 \; .5 \; 0 \; 0 \\ 0 \; .5 \; 0 \; .5 \; 0 \\ 0 \; 0 \; .5 \; 0 \; .5 \\ .5 \; 0 \; 0 \; .5 \; 0 \end{bmatrix}$$

(b)

$$P^5 = \begin{bmatrix} 0.0625 & 0.3125 & 0.1562 & 0.1562 & 0.3125 \\ 0.3125 & 0.0625 & 0.3125 & 0.1562 & 0.1562 \\ 0.1562 & 0.3125 & 0.0625 & 0.3125 & 0.1562 \\ 0.1562 & 0.1562 & 0.3125 & 0.0625 & 0.3125 \\ 0.3125 & 0.1562 & 0.1562 & 0.3125 & 0.0625 \end{bmatrix}$$

$$P^{10} = \begin{bmatrix} 0.2480 & 0.1611 & 0.2148 & 0.2148 & 0.1611 \\ 0.1611 & 0.2480 & 0.1611 & 0.2148 & 0.2148 \\ 0.2148 & 0.1611 & 0.2480 & 0.1611 & 0.2148 \\ 0.2148 & 0.2148 & 0.1611 & 0.2480 & 0.1611 \\ 0.1611 & 0.2148 & 0.2148 & 0.1611 & 0.2480 \end{bmatrix}$$

$$P^{20} = \begin{bmatrix} 0.2058 & 0.1953 & 0.2018 & 0.2018 & 0.1953 \\ 0.1953 & 0.2058 & 0.1953 & 0.2018 & 0.2018 \\ 0.2018 & 0.1953 & 0.2058 & 0.1953 & 0.2018 \\ 0.2018 & 0.2018 & 0.1953 & 0.2058 & 0.1953 \\ 0.1953 & 0.2018 & 0.2018 & 0.1953 & 0.2058 \end{bmatrix}$$

$$P^{40} = \begin{bmatrix} 0.2001 & 0.1999 & 0.2000 & 0.2000 & 0.1999 \\ 0.1999 & 0.2001 & 0.1999 & 0.2000 & 0.2000 \\ 0.2000 & 0.1999 & 0.2001 & 0.1999 & 0.2000 \\ 0.2000 & 0.2000 & 0.1999 & 0.2001 & 0.1999 \\ 0.1999 & 0.2000 & 0.2000 & 0.1999 & 0.2001 \end{bmatrix}$$

$$P^{80} = \begin{bmatrix} 0.2000 & 0.2000 & 0.2000 & 0.2000 & 0.2000 \\ 0.2000 & 0.2000 & 0.2000 & 0.2000 & 0.2000 \\ 0.2000 & 0.2000 & 0.2000 & 0.2000 & 0.2000 \\ 0.2000 & 0.2000 & 0.2000 & 0.2000 & 0.2000 \\ 0.2000 & 0.2000 & 0.2000 & 0.2000 & 0.2000 \end{bmatrix}$$

(c) The steady state probability vector $\pi = (\pi_0, \pi_1, \pi_2, \pi_3, \pi_4)$ satisfies the system of equations:

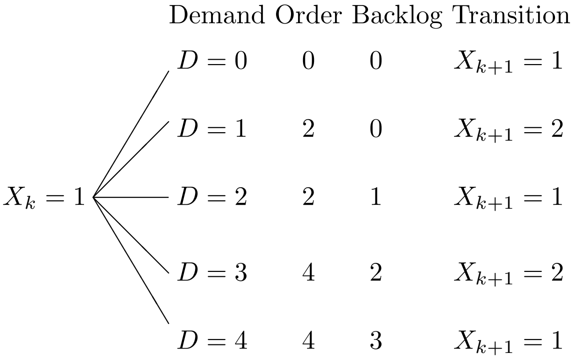
$$\pi P = \pi, \qquad \sum_{i=1}^{7} \pi_i = 1$$

which has solution

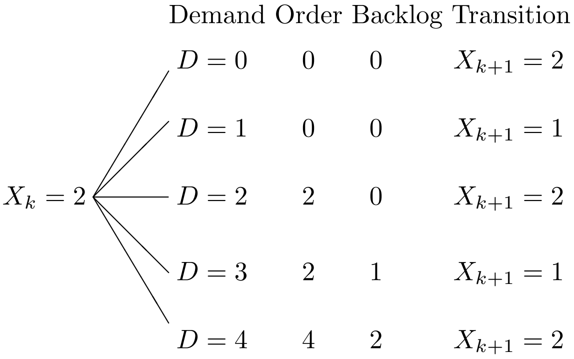
$$\pi = (\pi_0, \pi_1, \pi_2, \pi_3, \pi_4) = (0.2, 0.2, 0.2, 0.2, 0.2)$$

Clearly, this solution is the same as each of the row vectors of P^{80}, so our computations of P^n from part (b) show us how P^n converges to the matrix that has all its rows equal to the steady state probabilities.

2. (a) We first note that X_n = number of units in stock on hand after a delivery, has only two states, $\mathcal{S} = \{1, 2\}$. The following tree/table will be useful to compute the one-step transition matrix as well as answering part (d) of the question.



	Demand	Order	Backlog	Transition
$X_k = 1$	$D = 0$	0	0	$X_{k+1} = 1$
	$D = 1$	2	0	$X_{k+1} = 2$
	$D = 2$	2	1	$X_{k+1} = 1$
	$D = 3$	4	2	$X_{k+1} = 2$
	$D = 4$	4	3	$X_{k+1} = 1$



	Demand	Order	Backlog	Transition
$X_k = 2$	$D = 0$	0	0	$X_{k+1} = 2$
	$D = 1$	0	0	$X_{k+1} = 1$
	$D = 2$	2	0	$X_{k+1} = 2$
	$D = 3$	2	1	$X_{k+1} = 1$
	$D = 4$	4	2	$X_{k+1} = 2$

The transition matrix is then

$$P = \begin{bmatrix} 3/5 & 2/5 \\ 2/5 & 3/5 \end{bmatrix}$$

(b) The steady state probability vector $\pi = (\pi_1, \pi_2)$ satisfies

$$\begin{aligned} \frac{3}{5}\pi_1 + \frac{2}{5}\pi_2 &= \pi_1 \\ \frac{2}{5}\pi_1 + \frac{3}{5}\pi_2 &= \pi_2 \\ \pi_1 + \pi_2 &= 1 \end{aligned}$$

Substituting $\pi_1 = 1 - \pi_2$ in the first equation gives us

$$\frac{3}{5}(1-\pi_2) + \frac{2}{5}\pi_2 = 1 - \pi_2 \quad\Rightarrow\quad \frac{4}{5}\pi_2 = \frac{2}{5} \quad\Rightarrow\quad \pi_2 = \frac{1}{2} \quad\Rightarrow\quad \pi_1 = \frac{1}{2}$$

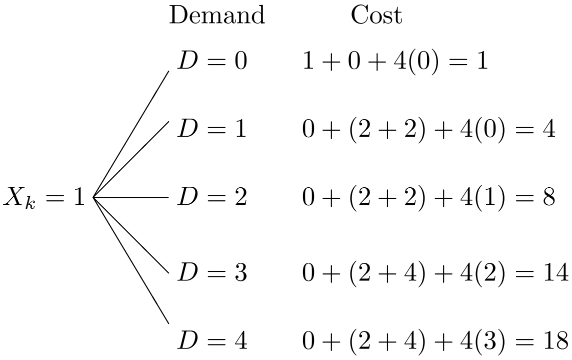
Therefore,

$$\pi = \left(\frac{1}{2}, \frac{1}{2}\right).$$

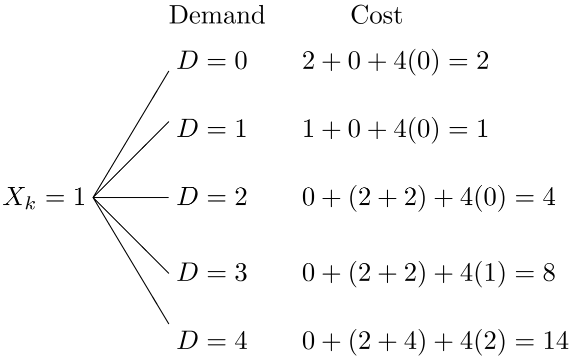
(c) Note that the cost at any particular time is given by

 # Units in storage + (2 + # Units ordered, if an order was placed) + 4·(# Units backlogged).

Using the trees from part (a) we can see that the cost for each of the two states depends on the demand and is given by



Demand	Cost
$D=0$	$1+0+4(0)=1$
$D=1$	$0+(2+2)+4(0)=4$
$D=2$	$0+(2+2)+4(1)=8$
$D=3$	$0+(2+4)+4(2)=14$
$D=4$	$0+(2+4)+4(3)=18$



Demand	Cost
$D=0$	$2+0+4(0)=2$
$D=1$	$1+0+4(0)=1$
$D=2$	$0+(2+2)+4(0)=4$
$D=3$	$0+(2+2)+4(1)=8$
$D=4$	$0+(2+4)+4(2)=14$

Since the demand on any time period is independent of X_n, and each possible value of the demand occurs with probability 1/5, then

$$\text{Expected average cost} = E[\text{Cost}, 1]\pi_1 + E[\text{Cost}, 2]\pi_2 = \frac{45}{5}\cdot\frac{1}{2} + \frac{29}{5}\cdot\frac{1}{2} = \frac{74}{10} = \frac{37}{5}$$

3. (a) For $i = c, w, s$ and $\rho = \frac{1}{1+r}$, let

$$u(i) = E_i \sum_{k=0}^{\infty} \rho^k f(X_k) = \sum_{k=0}^{\infty} \rho^k E_i[f(X_k)]$$

We are interested in computing $u(w)$, but in the process we will compute also $u(c), u(s)$. The basic technique is to analyze $E_i f(X_k)$ by conditioning on the first transition:

$$\begin{aligned}E_i[f(X_k)] &= P_{i,c}E[f(X_k)|X_0=i,X_1=c] + P_{i,w}E[f(X_k)|X_0=i,X_1=w]\\ &\quad + P_{i,s}E[f(X_k)|X_0=i,X_1=s]\\ &= P_{i,c}E_c[f(X_{k-1})] + P_{i,w}E_w[f(X_{k-1})] + P_{i,s}E_s[f(X_{k-1})]\end{aligned}$$

Plugging this expression into the formula for $u(i)$ gives

$$\begin{aligned}u(i) &= \sum_{k=0}^{\infty}\rho^k E_i[f(X_k)]\\ &= E_i[f(X_0)] + \sum_{k=1}^{\infty}\rho^k E_i[f(X_k)]\\ &= f(i) + \rho\sum_{k=1}^{\infty}\rho^{k-1}\left\{P_{i,c}E_c[f(X_{k-1})] + P_{i,w}E_w[f(X_{k-1})] + P_{i,s}E_s[f(X_{k-1})]\right\}\\ &= f(i) + \rho\left\{P_{i,c}\sum_{k=0}^{\infty}\rho^k E_c[f(X_k)] + P_{i,w}\sum_{k=0}^{\infty}\rho^k E_w[f(X_k)] + P_{i,s}\sum_{k=0}^{\infty}\rho^k E_s[f(X_k)]\right\}\\ &= f(i) + \rho\left\{P_{i,c}\,u(c) + P_{i,w}\,u(w) + P_{i,s}\,u(s)\right\}\end{aligned}$$

If we let $u = (u(c), u(w), u(s))^T$ and $f = (f(c), f(w), f(s))^T$, then in matrix notation the above simplifies to

$$u = f + \rho P u$$

which has as solution

$$u = (I - \rho P)^{-1} f$$

For $f = (-1, 3, 7)^T$, $\rho = \frac{1}{1.03}$, and $P = \begin{bmatrix}0.3 & 0.6 & 0.1\\ 0.1 & 0.8 & 0.1\\ 0.1 & 0.7 & 0.2\end{bmatrix}$,

$$(u(c), u(w), u(s)) = (96.2569, 101.2208, 105.6509)$$

So if this month is warm, the expected discounted profit is $u(w)$, which is greater than if this month was cold, $u(c)$, and smaller than if this month was scorching, $u(s)$.

(b) To compute the expected time to bankruptcy we are going to redefine the state space to be:

w : if the current month is warm
s : if the current month is scorching
c_1 : if the current month is cold but the previous one was not
c_2 : if the current and the previous month were cold, but the one before was not
b : if it has been cold three months in a row (bankrupt)

The new transition matrix for the modified Markov chain is

$$\hat{P} = \begin{bmatrix} 0.8 & 0.1 & 0.1 & 0 & 0 \\ 0.7 & 0.2 & 0.1 & 0 & 0 \\ 0.6 & 0.1 & 0 & 0.3 & 0 \\ 0.6 & 0.1 & 0 & 0 & 0.3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

We want to compute the expected time until bankruptcy assuming that this month was warm, that is, if

$$T = \text{time to bankruptcy}, \quad \text{and} \quad u(i) = E[T|X_0 = i], \quad i = w, s, c_1, c_2,$$

then we want to compute $u(w)$. By first transition analysis,

$$\begin{aligned} u(i) &= \hat{P}_{i,w}\, E[T|X_0 = i, X_1 = w] + \hat{P}_{i,s}\, E[T|X_0 = i, X_1 = s] + \hat{P}_{i,c_1}\, E[T|X_0 = i, X_1 = c_1] \\ &\quad + \hat{P}_{i,c_2}\, E[T|X_0 = i, X_1 = c_2] + \hat{P}_{i,b}\, E[T|X_0 = i, X_1 = b] \\ &= \hat{P}_{i,w}\,(1 + E[T|X_0 = w]) + \hat{P}_{i,s}\,(1 + E[T|X_0 = s]) + \hat{P}_{i,c_1}\,(1 + E[T|X_0 = c_1] \\ &\quad + \hat{P}_{i,c_2}\,(1 + E[T|X_0 = c_2]) + \hat{P}_{i,b}\,(1) \\ &= 1 + \hat{P}_{i,w}\, u(w) + \hat{P}_{i,s}\, u(s) + \hat{P}_{i,c_1}\, u(c_1) + \hat{P}_{i,c_2}\, u(c_2) \end{aligned}$$

By defining $u = (u(w), u(s), u(c_1), u(c_2))^T$, $e = (1, 1, 1, 1)^T$, and $\tilde{P} = \begin{bmatrix} 0.8 & 0.1 & 0.1 & 0 \\ 0.7 & 0.2 & 0.1 & 0 \\ 0.6 & 0.1 & 0 & 0.3 \\ 0.6 & 0.1 & 0 & 0 \end{bmatrix}$, we can rewrite the above equations as

$$u = e + \tilde{P}u, \quad \text{or equivalently,} \quad u = (I - \tilde{P})^{-1}e$$

(note that $\tilde{P}$ is substochastic, since its fourth row adds to less than one, so $(I - \tilde{P})$ is invertible). The solution is given by

$$(u(w), u(s), u(c_1), u(c_2)) = (125.5555, 125.5555, 115.5555, 88.8888)$$

(c) To compute the expected value of the total profits we follow the same approach as in part (b), with some minor modifications. Let

$$\tau = \text{time at which bankruptcy occurs}, \qquad u(i) = E_i \sum_{k=1}^{\tau-1} f(X_k), \quad i = w, s, c_1, c_2.$$

By conditioning on the first transition,

$$\begin{aligned}
u(i) &= \hat{P}_{i,w}\, E\left[\sum_{k=1}^{\tau-1} f(X_k) \,\middle|\, X_0 = i, X_1 = w\right] + \hat{P}_{i,s}\, E\left[\sum_{k=1}^{\tau-1} f(X_k) \,\middle|\, X_0 = i, X_1 = s\right] \\
&\quad + \hat{P}_{i,c_1}\, E\left[\sum_{k=1}^{\tau-1} f(X_k) \,\middle|\, X_0 = i, X_1 = c_1\right] + \hat{P}_{i,c_2}\, E\left[\sum_{k=1}^{\tau-1} f(X_k) \,\middle|\, X_0 = i, X_1 = c_2\right] \\
&\quad + \hat{P}_{i,b}\, E\left[\sum_{k=1}^{\tau-1} f(X_k) \,\middle|\, X_0 = i, X_1 = b\right] \\
&= \hat{P}_{i,w}\,(f(i) + u(w)) + \hat{P}_{i,s}\,(f(i) + u(s)) + \hat{P}_{i,c_1}\,(f(i) + u(c_1)) \\
&\quad + \hat{P}_{i,c_2}\,(f(i) + u(c_2)) + \hat{P}_{i,b}\, f(i) \\
&= f(i) + \hat{P}_{i,w}\, u(w) + \hat{P}_{i,s}\, u(s) + \hat{P}_{i,c_1}\, u(c_1) + \hat{P}_{i,c_2}\, u(c_2)
\end{aligned}$$

Define $f = (f(w), f(s), f(c), f(c))^T$ and let $\tilde{P}$ as before. Then the above simplifies to

$$u = f + \tilde{P}u, \quad \text{or equivalently,} \quad u = (I - \tilde{P})^{-1} f$$

The solution is given by

$$(u(w), u(s), u(c_1), u(c_2)) = (374.6913, 379.1358, 340.2469, 261.7284)$$