Assignment 4 - *Due Friday February 6th*

1. (H& L 16.3-3)[1] A particle moves on a circle through points that have been marked 0, 1, 2, 3, 4 (in a clockwise order). The particle starts at point 0. At each step it has probability 0.5 of moving one point clockwise (0 follows 4) and 0.5 of moving one point counterclockwise. let X_n ($n \geq 0$) denote its location on the circle after step n. $\{X_n\}$ is a Markov chain.

   (a) Construct the (one-step) transition matrix.

   (b) Calculate the n-step transition matrix $\mathbf{P}^{(n)}$ for $n = 5, 10, 20, 40, 80$.

   (c) Find the steady-state probabilities of the state of the Markov chain. Describe how the probabilities in the n-step transition matrices obtained in part (b) compare to these steady-state probabilities as n grows large.

2. (H&L 16.5-9) Consider the following inventory policy for a certain product. If the demand during a period exceeds the number of items available, this unsatisfied demand is backlogged; i.e., it is filled when the next order is received. Let Z_n ($n = 0, 1, \dots$) denote the amount of inventory on hand minus the number of units backlogged before ordering at the end of period n ($Z_0 = 0$). If Z_n is zero or positive, no orders are backlogged. If Z_n is negative, then $-Z_n$ represents the number of backlogged units and no inventory is on hand. At the end of period n, if $Z_n < 1$, an order is placed for $2m$ units, where m is the smallest integer such that $Z_n + 2m \geq 1$. Orders are filled immediately.

   Let $D_1, D_2, \dots,$ be the demand for a product in periods 1, 2, $\dots$, respectively. Assume that the D_n are independent and identically distributed random variables taking on the values, 0, 1, 2, 3, 4, each with probability $\frac{1}{5}$. Let X_n denote the amount of stock on hand *after* ordering at the end of period n (where $X_0 = 2$), so that

$$X_n = \begin{cases} X_{n-1} - D_n + 2m & \text{if } X_{n-1} - D_n < 1 \\ X_{n-1} - D_n & \text{if } X_{n-1} - D_n \geq 1 \end{cases} \qquad (n = 1, 2, \dots),$$

   when $\{X_n\}$ ($n = 0, 1, \dots$) is a Markov chain. It has only two states, 1 and 2, because the only time that ordering will take place is when $Z_n = 0, -1, -2,$ or -3, in which case 2, 2, 4, and 4 units are ordered, respectively, leaving $X_n = 2, 1, 2, 1$, respectively.

   (a) Construct the (one-step) transition matrix.

   (b) Use the steady-state equations to solve manually for the steady-state probabilities.

   (c) Suppose that the ordering cost is given by $(2 + 2m)$ if an order is placed and zero otherwise. The holding cost per period is Z_n if $Z_n \geq 0$ and zero otherwise. The shortage cost per period is $-4Z_n$ if $Z_n < 0$ and zero otherwise. Find the (long-run) expected average cost per unit time.

[1]Most of the problems in this assignment are from Hillier & Lieberman *Introduction to Operations Research*, 7th edition.

3. Suppose that the monthly profits for a beer manufacturer depend on the average temperature during the month. The average temperature in a given month can take 3 values: *cold (c), warm (w),* and *scorching (s).* The monthly profit equals to $-1[M\$]$ in a cold month, $3[M\$]$ in a warm month and $7[M\$]$ in a scorching month. Assume that the average monthly temperature evolves according to a Markov chain with one-step transition matrix P given by:

$$P = \begin{array}{c} \\ C \\ W \\ S \end{array} \begin{array}{c} \begin{array}{ccc} C & W & S \end{array} \\ \begin{bmatrix} 0.3 & 0.6 & 0.1 \\ 0.1 & 0.8 & 0.1 \\ 0.1 & 0.7 & 0.2 \end{bmatrix} \end{array}$$

   (a) Compute the expected infinite horizon discounted profit, with a discount rate $r = 3\%$, and assuming that this month is warm. That is, compute

$$E_w \sum_{k=0}^{\infty} \left(\frac{1}{1+r}\right)^k f(X_k),$$

   where $f(c) = -1$, $f(w) = 3$ and $f(s) = 7$. How does your result change if this month is scorching? And if cold?

Suppose that the company will go bankrupt when there are 3 consecutive cold months.

   (b) Compute the expected time until bankruptcy assuming that the average temperature this month is warm.

   (c) Compute the expected value of the total profits accrued prior to bankruptcy.