Assignment 1 Solution

1. (a) The mean of X is given by

$$\mathrm{E}X = \int_0^\infty x \frac{\lambda(\lambda x)^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)}\, dx = \frac{1}{\lambda\Gamma(\alpha)} \int_0^\infty \lambda(\lambda x)^{\alpha} e^{-\lambda x}\, dx = \frac{\Gamma(\alpha+1)}{\lambda\Gamma(\alpha)} = \frac{\alpha}{\lambda}$$

Similarly, we have

$$\mathrm{E}X^2 = \int_0^\infty x^2 \frac{\lambda(\lambda x)^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)}\, dx = \frac{1}{\lambda^2\Gamma(\alpha)} \int_0^\infty \lambda(\lambda x)^{\alpha+1} e^{-\lambda x}\, dx = \frac{\Gamma(\alpha+2)}{\lambda\Gamma(\alpha)} = \frac{(\alpha+1)\alpha}{\lambda},$$

so

$$\mathrm{Var}(X) = \mathrm{E}X^2 - (\mathrm{E}X)^2 = \frac{\alpha}{\lambda^2}.$$

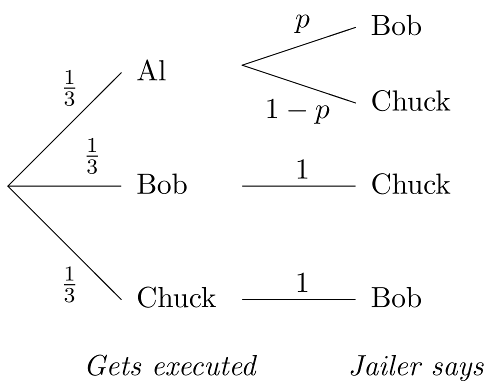
(b) For $n = 1$, $S_1 = X_1$ has exponential distribution with pdf $f_{S_1}(x) = \lambda e^{-\lambda x}, x \geq 0$. Noting $\Gamma(1) = 1$, we conclude that S_1 has Gamma distribution with scale parameter λ and shape parameter $\alpha = 1$.

To apply induction, suppose S_n has Gamma distribution with scale parameter λ and shape parameter $\alpha = n$. Then, $S_{n+1} = S_n + X_{n+1}$ has pdf $f_{S_{n+1}}(x)$ given by

$$\begin{aligned} f_{S_{n+1}}(x) &= \int_{-\infty}^{\infty} f_{S_n}(y) f_{X_{n+1}}(x-y)\, dy \\ &= \int_0^x f_{S_n}(y) f_{X_{n+1}}(x-y)\, dy \\ &= \int_0^x \frac{\lambda(\lambda y)^{n-1} e^{-\lambda y}}{\Gamma(n)} \cdot \lambda e^{-\lambda(x-y)}\, dy \\ &= \frac{\lambda^{n+1} e^{-\lambda x}}{\Gamma(n)} \int_0^x y^{n-1}\, dy \\ &= \frac{\lambda(\lambda x)^n e^{-\lambda x}}{\Gamma(n+1)} \end{aligned}$$

so S_{n+1} has Gamma distribution with scale parameter λ and shape parameter $\alpha = n+1$, finishing the induction.

2. The information in the problem can be represented by the following tree, where p is the probability that the jailer will tell Al that Bob is being freed.



Let X be the prisoner who will get executed, and let Y the name the jailer tells Al will be freed. Then, by Baye's formula, $P(X = \text{Al}|Y = \text{Bob})$ equals

$$\frac{P(Y = \text{B}|X = \text{A})P(X = \text{A})}{P(Y = \text{B}|X = \text{A})P(X = \text{A}) + P(Y = \text{B}|X = \text{B})P(X = \text{B}) + P(Y = \text{B}|X = \text{C})P(X = \text{C})}$$

From Al's perspective,

$$P(X = \text{A}) = P(X = \text{B}) = P(X = \text{C}) = \frac{1}{3}$$

and

$$P(Y = \text{B}|X = \text{A}) = p, P(Y = \text{B}|X = \text{B}) = 0, P(Y = \text{B}|X = \text{C}) = 1$$

Hence,

$$P(Y = \text{B}|X = \text{A} = \frac{p/3}{p/3 + 1/3} = \frac{p}{p+1}$$

(a) In this case, $p = \frac{1}{2}$, then

$$P(X = \text{Al}|Y = \text{Bob}) = \frac{1}{3}$$

(b) In this case, $p = 1$, then

$$P(X = \text{Al}|Y = \text{Bob}) = \frac{1}{2}$$

(c) In this case, $p = q$, then

$$P(X = \text{Al}|Y = \text{Bob}) = \frac{q}{q+1}$$

3. (a) If A loses , he will gain \$$(-x)$; if he wins, he will gain \$$(1 - x)$. Hence, the expected gain of A in one game is $G \triangleq p(1-x) - (1-p)x$ which must be zero, since otherwise the expected gain in n games would be nG which goes to ∞ (in which case B won't play) or $-\infty$ (in which case A won't play). It follows that $p = x$.

   (b) In the spirit of part (A), it suffices that we find the probability that A first win three games (so that he wins the "entire game"). Clearly ,they will determine the winner in three, four or five rounds. If they are three rounds in total, then A must win first three games in a row and this occurs with probability p^3; if there are four rounds in total, then A must win the fourth round and win two rounds out of the first three rounds, so A wins after four rounds with probability $\binom{3}{2}p^3(1-p)$; if there are five rounds in total, then A must win the fifth round and win two rounds out of the first four rounds, so A wins after five rounds with probability $\binom{4}{2}p^3(1-p)^2$. Therefore, the probability that A wins the \$1 pot is

$$p^3 + \binom{3}{2}p^3(1-p) + \binom{4}{2}p^3(1-p)^2 = p^3(6p^2 - 15p + 10)$$

   which is also the money that A should put in the pot.

4. (a) Suppose we follow this strategy and we lose n ($n \geq 0$) consecutive rounds before the first win. Then the total loss of the first n rounds is given by

$$1 + 2 + \cdots + 2^{n-1} = 2^n - 1.$$

Note that the wager we put in the $n+1$ round is 2^n, so the total amount of profit we make is $2^n - (2^n - 1) = 1$ if we win the $n+1$ round. Hence, $W = 1$ with probability 1.

(b) From the analysis in part (a), it follows that $L = 2^n - 1, n = 0, 1, 2, \ldots$ and the probability distribution of L is given by

$$P(L = 2^n - 1) = P(\text{we didn't win until the } n+1 \text{ round }) = \left(\frac{1}{2}\right)^n \cdot \frac{1}{2} = \frac{1}{2^{n+1}}.$$

(c) The mean of L is

$$\mathrm{E}L = \sum_{n=0}^{\infty} (2^n - 1) \cdot P(L = 2^n - 1) = \sum_{n=0}^{\infty} \frac{2^n - 1}{2^{n+1}} = \sum_{n=0}^{\infty} \left(\frac{1}{2} - \frac{1}{2^{n+1}}\right) = \infty.$$

(d) It is obviously not a practical strategy because no matter how much money we have, we expect to lose all of them before we actually win one round to obtain \$1 profit.

5. The states of nature for this experiment are: F_1={the child has TB}, and F_2 = {the child does not have TB}. The possible outcomes of the test are: E = {positive test result}, and E^C = {negative test result}. From the problem description, we know that $P(E|F_1) = 0.90$ and $P(E|F_2) = 0.01$. Applying Bayes' Rule, we obtain the appropriate conditional probabilities.

(a) The probability that the child actually has TB given that she has tested positive in the Manoux test is $P(F_1|E)$.

$$\begin{aligned} P(F_1|E) &= \frac{P(E|F_1)P(F_1)}{P(E|F_1)P(F_1) + P(E|F_2)P(F_2)} \\ &= \frac{(0.90)(0.005)}{(0.90)(0.005) + (0.01)(1 - 0.005)} \\ &= 0.31. \end{aligned}$$

(b) The probability that the child actually has TB given that she has tested negative in the Manoux test is $P(F_1|E^C)$.

$$\begin{aligned} P(F_1|E^C) &= \frac{P(E^C|F_1)P(F_1)}{P(E^C|F_1)P(F_1) + P(E^C|F_2)P(F_2)} \\ &= \frac{(1 - 0.90)(0.005)}{(1 - 0.90)(0.005) + (1 - 0.01)(1 - 0.005)} \\ &= 0.0005. \end{aligned}$$

(c) Sensitivity Analysis.

- By increasing the likelihood of a true positive to $P(E|F_1) = 0.99$, our previous results are $P(F_1|E) = 0.33$ and $P(F_1|E^C) = 0.00005$.
- By decreasing the likelihood of a false positive to $P(E|F_2) = 0.001$, our previous results are $P(F_1|E) = 0.82$ and $P(F_1|E^C) = 0.0005$.

So our manufacturer of the Manoux test, it would make sense to try to reduce the likelihood of a false positive.