Assignment 1 - *Due Friday January 16th*

1. (a) The function defined by
$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\, dx$$
is called the "gamma function", and is a well-known "tabulated function" (just like the cumulative distribution function for a normal rv is a tabulated function).
We say that a rv X has a gamma distribution with (scale) parameter $\lambda > 0$ and (shape) parameter $\alpha > 0$ if its probability density function is given by
$$f_X(x) = \begin{cases} \frac{\lambda(\lambda x)^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)}, & x \geq 0 \\ 0, & x < 0. \end{cases}$$
Compute the mean and variance of X in terms of the gamma function.

   (b) Suppose that S_n is the time at which the n'th order is placed at a production facility. We represent S_n as $S_n = X_1 + \cdots + X_n$ (so that X_i is the "inter-arrival" time" between order $i-1$ and order i).
Assume that $X_1, X_2, \ldots$ is a sequence of iid exponential rv's with parameter $\lambda > 0$. Show that S_n has a gamma distribution and compute its parameters.

2. (Ross, Ex. 1.46) Three prisoners, named Al, Bob, and Chuck, are informed by their jailer that one of them has been chosen at random to be executed, and the other two are to be freed. Al asks the jailer to tell him privately which of his fellow prisoners will be set free, claiming that there would be no harm in divulging this information, since he already knows that at least one will go free.

   (a) Assume that if both Bob and Chuck will b e set free, then the jailer flips a fair coin to decide which name to tell Al. Suppose the jailer tells Al that Bob will be set free. What is the probability that Al will be executed?

   (b) Assume that if both Bob and Chuck will be set free , then the jailer (a firm believer in proper alphabetical ordering) will pick Bob's name to tell Al. Suppose the jailer tells Al that Bob will set free. What is the probability that Al will be executed?

   (c) Assume that if both Bob and Chuck will be set free, then the jailer (happy owner of a biased - Euro? - coin, picks Bob's name with probability q and Chuck' s name with probability $1-q$. Suppose the jailer tells Al that Bob will be set free. What is the probability that Al will be executed?

3. (a) Suppose that Player A and Player B play a game of chance in which either Player A or Player B will win the \$1 pot (but not both). In view of their skill levels, they agree that Player A should put $\$x$ in the pot and Player B $\$(1-x)$. What does this imply about their belief that Player A will win the game? (In other words, argue that this implies that the two players agree that Player A has a probability p of winning the game, where p can be computed in terms of x. Hint: Consider what should happen if they repeated this game of chance many times.)

(b) Now, suppose that Player A and Player B decide that the pot of $1 will go to the first player to win three games. How much money should Player A be expected to put in the pot?

4. Suppose that you are making successive wagers at a roulette wheel on which alternating slots are colored red and black, so that each color appears independently on each spin of the wheel with probability 1/2. You decide that your betting strategy will be to start with a one dollar bet on black, and to continue doubling your wager on each successive spin until you win.

   (a) Let W be the total amount that you will win if you follow this strategy. Compute the distribution of W.

   (b) Let L be the maximum loss before winning (i.e. the amount of money you have lost before the spin on which you win). Compute the distribution of L.

   (c) Compute the expectation of L.

   (d) Is this a practical strategy to follow?

5. Suppose that 0.5% of all children have tuberculosis (TB). When a child that has TB is given the Mantoux test, a positive result occurs 90% of the time. When a child who does not have TB is given the Mantoux test, a (false) positive result occurs 1% of the time.

   (a) Given that a child is tested and a positive result occurs, what is the probability that the child actually has TB?

   (b) Given that a child is tested and a negative result occurs, what is the probability that the child actually has TB?

   (c) As the manufacture of the Mantoux test, if you could spend (the same) money to either

      - increase the likelihood of a true positive result from 90% to 99%
      - decrease the likelihood of a false positive result from 1% to 0.1%

      which would you prefer and why?