Heavy Traffic Approximations

For a single-server queue, one can find good approximations to the *distribution* of time-in-system, number-in-system, etc. when $\rho =$ utilization $\left(= \frac{E[\text{service time}]}{E[\text{interarrival time}]} \right)$ is close to one (but, obviously, less than one!). Such approximations are called "heavy-traffic approximations".

Approximation 1 (Time-in-Queue)

When $\rho \approx 1$, then

$$\mathcal{W}_q \overset{D}{\approx} \frac{(\sigma_A^2 + \sigma_S^2)}{2(m_A - m_S)} \, exp(1)$$

where

$$\begin{aligned} \sigma_A^2 &= \text{variance of inter-arrival time} \\ \sigma_S^2 &= \text{variance of service time} \\ m_A &= \text{mean inter-arrival time} \\ m_S &= \text{mean service time} \\ exp(1) &= \text{exponential r.v. having mean 1} \end{aligned}$$

Approximation 2 (Number-in-Queue)

When $\rho \approx 1$, then

$$N_q \overset{D}{\approx} \frac{(\sigma_A^2 + \sigma_S^2)}{2m_S(m_A - m_S)} \, exp(1)$$

Approximation 3 (Time-in-System)

When $\rho \approx 1$, then

$$\mathcal{W} \overset{D}{\approx} \frac{(\sigma_A^2 + \sigma_S^2)}{2(m_A - m_S)} \, exp(1) + S$$

where S is a service time r.v. that is independent of the exponential.

Approximation 4 (Number-in-Queue)

When $\rho \approx 1$, then

$$N \overset{D}{\approx} \frac{(\sigma_A^2 + \sigma_S^2)}{2m_S(m_A - m_S)} \, exp(1) + \rho$$

Example:

Consider a processing plant where the inter-arrival time between orders is Erlang distributed with shape parameter 7 and a mean of 3 days. The time required to process an order is uniformly distributed between 0 and 5 days.

1. What is the probability that an order takes longer than 5 days?

   Note that $\sigma_A^2 = 9/7$, $\sigma_S^2 = 25/12$, $m_A = 3$, $m_S = 5/2$, and $\rho = m_S/m_A = 5/6$. Let S be a Uniform (0, 5) random variable, representing one service time. We need to compute $P(\mathcal{W} > 5)$, and since ρ is close to 1 we can approximate it with:

$$\begin{aligned}
P(\mathcal{W} > 5) &\approx P\left(\frac{(\sigma_A^2 + \sigma_S^2)}{2(m_A - m_S)}\, exp(1) + S > 5\right) \\
&= P\left(\frac{283}{84}\, exp(1) + S > 5\right) \\
&= \int_0^5 P\left(exp(1) > (5 - S)\frac{84}{283}\middle| S = s\right)\frac{1}{5}ds \\
&= \frac{1}{5}\int_0^5 P\left(exp(1) > (5 - s)\frac{84}{283}\right)ds \\
&= \frac{1}{5}\int_0^5 e^{-(5-s)\frac{84}{283}}ds \\
&= \frac{e^{-\frac{240}{283}}}{5}\int_0^5 e^{\frac{84}{283}s}ds \\
&= \frac{283}{420}(1 - e^{-\frac{420}{283}}) \\
&= 0.521
\end{aligned}$$

2. What is the probability that the order book (the number of unfulfilled orders) has more than 8 orders in it?

   We need to compute $P(N > 8)$, and since $\rho = 14/15$ is close to 1, we can approximate it with:

$$\begin{aligned}
P(N > 8) &\approx P\left(\frac{(\sigma_A^2 + \sigma_S^2)}{2m_S(m_A - m_S)}\, exp(1) + \rho > 8\right) \\
&= P\left(\frac{283}{210}\, exp(1) + \frac{5}{6} > 8\right) \\
&= P\left(exp(1) > \frac{1505}{283}\right) \\
&= e^{-\frac{1505}{283}} \\
&= 0.0049
\end{aligned}$$