Midterm Exam

Place all answers on the question sheet provided. The exam is open textbook (Hillier and Lieberman) and open notes/handouts/homework. Write all answers clearly and in complete sentences. All answers should be supported by analysis or an argument.

First Name: ______________________________

Last Name: ______________________________

1(a)	1(b)	1(c)	1(d)	1(e)	1(f)	1(g)	2(a)	2(b)	3 (a)	3(b)	3(c)

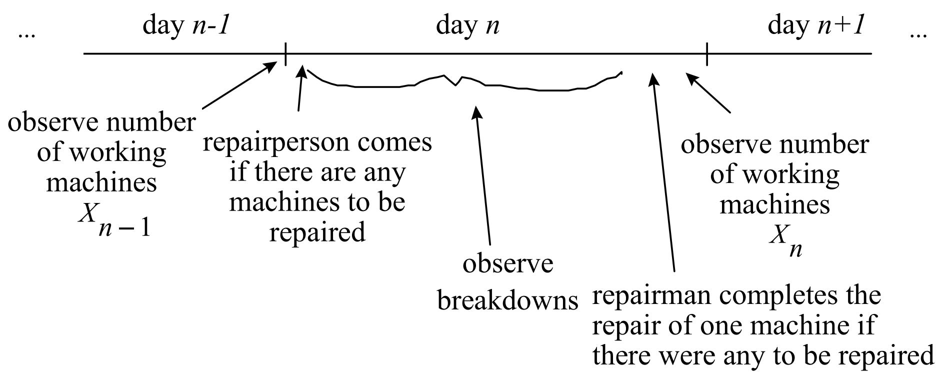
The Stanford Honor Code

1. The Honor Code is an undertaking of the students, individually and collectively:
   (a) That they will not give or receive aid in examinations; that they will not give or receive unpermitted aid in class work, in preparation of reports, or in any other work that is to be used by the instructor as the basis of grading;
   (b) that they will do their share and take an active part in seeing to it that others as well as themselves uphold the spirit and letter of the Honor Code.
2. The faculty on its part manifests its confidence in the honor of its students by refraining from proctoring examinations and from taking unusual and unreasonable precautions to prevent the forms of dishonesty mentioned above. The faculty will also avoid, as far as practicable, academic procedures that create temptations to violate the Honor Code.
3. While the faculty alone has the right and obligation to set academic requirements, the students and faculty will work together to establish optimal conditions for honorable academic work. I acknowledge and accept the Honor Code.

Signature: ______________________________

Question 1 (22 POINTS)

The MS&E department has 2 copy machines. On any given day, each working copy machine breaks down with probability 1/5 independent of the other machine. At the end of each day, if any machines are broken, a repairperson is called. The repairperson comes the following morning and can work only on one machine that day. When the repairperson has one or more machines to repair at the beginning of a day, he repairs exactly one. It is repaired by the end of the day but not used until the following day. Let X_n be the number of working machines at the end of day n, $n = 0, 1, 2, \ldots$ after breakdowns and repair that day are included. The event sequence is summarized below.



(a) [2 PTS] What is the state space of the Markov chain $X_0, X_1, X_2, \ldots$?

Solution: State space is 0, 1, 2, representing the number of working machines at the end of each day.

(b) [4 PTS] Let Y_n be a random variable denoting the number of breakdowns in day n. Clearly Y_n is dependent on the number X_{n-1} working at the end of day $n-1$. Describe the distribution function of Y_n for each possible value of X_{n-1}.

Solution:

- If $X_{n-1} = 0$, $P(Y_n = 0) = 1$
- If $X_{n-1} = 1$, $P(Y_n = 0) = 4/5$, $P(Y_n = 1) = 1/5$
- If $X_{n-1} = 2$, $P(Y_n = 0) = 16/25$, $P(Y_n = 1) = 8/25$, $P(Y_n = 2) = 1/25$

(c) [3 PTS] Compute the one-step transition probability matrix P for the Markov chain.

Solution:

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1/5 & 4/5 \\ 1/25 & 8/25 & 16/25 \end{bmatrix}$$

(d) [2 PTS] What is the period of the Markov chain?

Solution: All states are aperiodic. This follows because all states communicate, and therefore have the same period. Since $P(1 \to 1) > 0$, state 1 is aperiodic.

(e) [3 PTS] Suppose that currently only one copier is operational. What is the probability that neither copier will be operational the day after tomorrow?

Solution:

$$P(X_2 = 0|X_0 = 1) = (P^2)(1, 0) = 4/125$$

(f) [4 PTS] Compute the steady state distribution of the number of copiers working at the end of a day. (Fractions are fine; do not bother to compute decimal quantities).

Solution: Let π be the steady state distribution. We need to solve

$$\begin{cases} \pi_0 = \frac{1}{25}\pi_2 \\ \pi_1 = \pi_0 + \frac{1}{5}\pi_1 + \frac{8}{25}\pi_2 \\ \pi_2 = \frac{4}{5}\pi_1 + \frac{16}{25}\pi_2 \\ \pi_0 + \pi_1 + \pi_2 = 1 \end{cases}$$

The solution is $\pi_0 = 4/149$, $\pi_1 = 45/149$, $\pi_2 = 100/149$.

(g) [4 PTS] The repairperson earns $40 for each day of work. What is his or her long run expected income per day?

Solution: The repairperson works on the days in which there are fewer than 2 working machines (states 0 and 1). His long run expected daily income is therefore

$$40(\pi_0 + \pi_1) = \frac{1960}{149}$$

Question 2 (7 POINTS)

(a) [3 PTS] We need to implement a simulation in which the single-period return X has the distribution

$$F_X(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{96}x^5 + \frac{1}{3}x, & 0 \le x \le 2 \\ 1, & x > 2 \end{cases}$$

Suppose we have a sequence of iid Uniform (0,1) rvs U_1, U_2,.... Describe your algorithm for generating the rv X.

Solution: We use the acceptanceCrejection algorithm. Since the density function $f_X(x)$ of X is given by

$$f_X(x) = \frac{d}{dx}F_X(x) = \begin{cases} \frac{5}{96}x^4 + \frac{1}{3}, & 0 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$$

and so

$$M = \max_{0 \le x \le 2} f_X(x) = f_X(2) = \frac{7}{6}$$

i) Generate U_1, U_2 iid $\sim$ Unif[0,1].

ii) If $MU_2 \le f_X(2U_1)$, i.e., $7U_2 \le 5U_1^4 + 2$, then return $X = 2U_1$. Otherwise, return to i).

(b) [4 PTS] Suppose that we use the simulation to compute the probability p that the total return over the next ten years is greater than 200%. We run the simulation 100 independent times and find that 91 of the simulations have returns greater than 200%. Provide a 95% confidence interval for p. Note for a Normal (0,1) rv,

$$P(N(0,1) \le 1.648) = 0.95 \qquad P(N(0,1) \le 1.96) = 0.975$$
$$P(N(0,1) \le 2.33) = 0.99 \qquad P(N(0,1) \le 2.575) = 0.995$$

Solution: Let Y_n be 1 if the nth simulation returns greater than 200%, and be 0 otherwise. Let $\widehat{Y}$ and s be the sample mean and sample deviation, respectively. Then,

$$\widehat{Y} = \frac{1}{100}\sum_{n=1}^{100} Y_n = \frac{91}{100}$$

$$s^2 = \frac{1}{99}\sum_{n=1}^{100}(Y_n - \widehat{Y})^2 = \frac{1}{99}[91 \times (1 - 0.91)^2 + 9 \times (0 - 0.91)^2] \approx 0.0827$$

so the 95% confidence interval is $[0.91 - 1.96s/10, 0.91 + 1.96s/10]$, i.e. $[0.8536, 0.9664]$.

Question 3 (11 POINTS)

(a) [4 PTS] Suppose you own a speculative stock and you observe the stock price of the past ten days (excluding today) are $Y_0 = 4$, $Y_1 = 5$, $Y_2 = 3$, $Y_3 = 3$, $Y_4 = 6$, $Y_5 = 7$, $Y_6 = 6$, $Y_7 = 8$, $Y_8 = 6$, $Y_9 = 5$. Fit first order autoregressive model to this time series (Compute the parameters). You might need the following numbers.

$$\sum_{i=0}^{9} Y_i^2 = 305 \qquad \sum_{i=1}^{9} Y_i = 49 \qquad \sum_{i=1}^{9} Y_i Y_{i-1} = 272$$

$$\sum_{i=0}^{8} Y_i^2 = 280 \qquad \sum_{i=0}^{8} Y_i = 48 \qquad \sum_{i=1}^{9} Y_i^2 Y_{i-1} = 1648$$

Set up the equations (but do not solve them) for fitting the parameters.

Solution:

$$\hat{a} = \frac{\sum_{i=1}^{9} Y_i Y_{i-1} - 9(\sum_{i=1}^{9} Y_i/9)(\sum_{i=0}^{8} Y_i/9)}{\sum_{i=0}^{8} Y_i^2 - 9(\sum_{i=1}^{9} Y_i/9)(\sum_{i=0}^{8} Y_i/9)} = \frac{272 - 9 \cdot (49/9) \cdot (48/9)}{280 - 9 \cdot (49/9) \cdot (48/9)} = \frac{4}{7}$$

$$\hat{b} = \frac{1}{9}\sum_{i=1}^{9} Y_i - \frac{1}{9}\hat{a}\sum_{i=0}^{8} Y_i = \frac{49}{9} - \frac{1}{9} \cdot \frac{4}{7} \cdot 48 = \frac{151}{63}$$

(b) [4 PTS] Suppose that we believe that the third order autoregressive process

$$Y_{n+1} = \frac{1}{2}Y_n - \frac{2}{5}Y_{n-1} + \frac{1}{5}Y_{n-2} + 4 + \epsilon_{n+1}$$

(with the ϵ_i's iid $N(0,1)$ rvs) fits the data well. Predict Y_{10}, Y_{11} and Y_{12} using this model.

Solution:

$$\hat{Y}_{10} = (1,0,0)\begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} + (1,0,0)\begin{pmatrix} 1/2 & -2/5 & 1/5 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 5 \\ 6 \\ 8 \end{pmatrix} = 5.7$$

$$\hat{Y}_{11} = (1,0,0)\sum_{j=0}^{1}\begin{pmatrix} 1/2 & -2/5 & 1/5 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^j\begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} + (1,0,0)\begin{pmatrix} 1/2 & -2/5 & 1/5 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^2\begin{pmatrix} 5 \\ 6 \\ 8 \end{pmatrix} = 6.05$$

$$\hat{Y}_{12} = (1,0,0)\sum_{j=0}^{2}\begin{pmatrix} 1/2 & -2/5 & 1/5 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^j\begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} + (1,0,0)\begin{pmatrix} 1/2 & -2/5 & 1/5 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^3\begin{pmatrix} 5 \\ 6 \\ 8 \end{pmatrix} = 5.745$$

(c) [3 PTS] Assuming the model of part (b), what is the probability that Y_{12} is greater than 7?

Solution:

$$\begin{aligned} Y_{12} &= (1,0,0)\sum_{j=0}^{2}\begin{pmatrix} 1/2 & -2/5 & 1/5 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^j\begin{pmatrix} 4+\epsilon_{12-j} \\ 0 \\ 0 \end{pmatrix} + (1,0,0)\begin{pmatrix} 1/2 & -2/5 & 1/5 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^3\begin{pmatrix} 5 \\ 6 \\ 8 \end{pmatrix} \\ &= 5.745 + \epsilon_{12} + 0.5\epsilon_{11} - 0.15\epsilon_{10} \end{aligned}$$

Note that

$$\epsilon_{12} + 0.5\epsilon_{11} - 0.15\epsilon_{10} \stackrel{\mathfrak{D}}{=} N(0,1) + N(0,0.25) + N(0,0.0225) \stackrel{\mathfrak{D}}{=} N(0,1.2725)$$

So,

$$P(Y_{12} > 7) = P(N(0,1.2725) > 1.255) = P(N(0,1) > 1.255/\sqrt{1.2725}) = 0.133$$