Final Exam

Place all answers on the question sheet provided. The exam is open textbook (Hillier and Lieberman) and open notes/handouts/homework. Write all answers clearly and in complete sentences. All answers should be supported by analysis or an argument.

First Name: ______________________

Last Name: ______________________

1(a)	1(b)	1(c)	1(d)	1(e)	2(a)	2(b)	2(c)	2(d)	3(a)

3(b)	3(c)	4(a)	4(b)	4(c)	4(d)	5(a)	5(b)	5(c)	Total

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(2) That they will do their share and take an active part in seeing to it that others as well as themselves uphold the spirit and letter of the Honor Code.

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C. While the faculty alone has the right and obligation to set academic requirements, the students and faculty will work together to establish optimal conditions for honorable academic work.

I acknowledge and accept the Honor Code.

Signature: ______________________

Question 1 (21 POINTS)

In this problem, we will look at several different design issues that arise in the context of service-based systems.

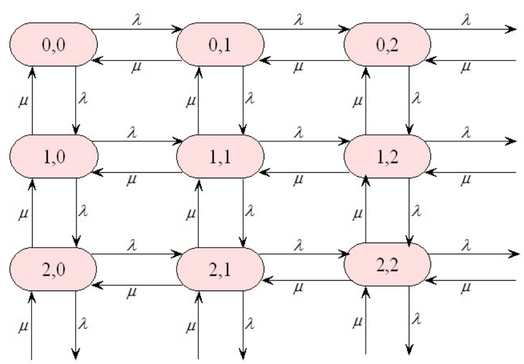
(a) [3 PTS] Suppose that a facility currently has two service units, each of which is fed by its own Poisson arrival process and processes its own individual waiting room. Each Poisson process has an arrival rate of 5 customers per hour (for a total arrival rate of 10 customers per hour). Each arriving customer's service time is exponentially distributed with a mean of 6 minutes, and each service unit processes work at unit rate (i.e., one unit of service time per unit of "wall clock" time). What is the average time-in-system for a customer?

Solution: Since the two queues are identical, we only need to consider one of them, i.e., a M/M/1 queue with arrival rate $\lambda = 5$ and service rate $\mu = 10$. So the average time-in-system is

$$W = \frac{1}{\mu - \lambda} = \frac{1}{5}(\text{hrs}) = 12(\text{min})$$

(b) [4 PTS] For the model in (a), suppose that there are currently 3 customers at unit 1 and 4 customers at unit 2. Write down a system of linear equations (but do not solve) from which you can compute the expected time until both servers are simultaneously idle.

Solution: Let $X(t)$ and $Y(t)$ be the number of customers in unit 1 and 2 at time t, respectively. Then $\{(X(t), Y(t) : t \geq 0\}$ forms a CTMC with the following diagram:



Let $u(n, m)$ be the expected time until both servers are simultaneously idle if there are currently n customers at unit 1 and m customers at unit 2. Then we have the following linear system:

$$u(n,m) = \frac{1}{2(\mu+\lambda)} + \frac{\mu}{2(\mu+\lambda)}(u(n-1,m)+u(n,m-1)) + \frac{\lambda}{2(\mu+\lambda)}(u(n+1,m)+u(n,m+1))$$
$$u(0,m) = \frac{1}{\mu+2\lambda} + \frac{\mu}{\mu+2\lambda}u(0,m-1) + \frac{\lambda}{\mu+2\lambda}(u(1,m)+u(0,m+1))$$
$$u(n,0) = \frac{1}{\mu+2\lambda} + \frac{\mu}{\mu+2\lambda}u(n-1,0) + \frac{\lambda}{\mu+2\lambda}(u(n,1)+u(n+1,0))$$

for $n, m \geq 1$ and we have boundary condition $u(0,0) = 0$.

(c) [4 PTS] Now suppose that we combine the two service units. There is only one queue that forms (that is fed by the combined superposition of the two arrival streams), and the facility has two servers that each serves at unit rate. Customers are assigned to the first available server. What is the average time-in-system for a customer in this design?

<u>**Solution:**</u> Now we have a M/M/2 queue with arrival rate $\lambda = 10$ and service rate $\mu = 10$ for each server. Let $\rho = \lambda/(2\mu) = 1/2$ be the utilization factor of the system. Then, the steady-state distribution for no customers in the system is

$$\pi_0 = 1/\left[\sum_{n=0}^{s-1} \frac{(\lambda/\mu)^n}{n!} + \frac{(\lambda/\mu)^s}{s!}\frac{1}{1-\rho}\right] = 1/\left[1 + 1 + \frac{1}{2}\cdot\frac{1}{1-1/2}\right] = \frac{9}{4}$$

where $s = 2$. Further, the average queue length is

$$L_q = \frac{\pi_0(\lambda/\mu)^s\rho}{s!(1-\rho)^2} = \frac{\frac{9}{4}\cdot 1\cdot\frac{1}{2}}{2\cdot\left(1-\frac{1}{2}\right)^2} = \frac{9}{4}$$

so the average time-in-system is

$$W = \frac{L_q}{\lambda} + \frac{1}{\mu} = \frac{9}{4}\cdot\frac{1}{10} + \frac{1}{10} = \frac{13}{40}(\text{hrs}) = 19.5(\text{mins}).$$

(d) [4 PTS] Now suppose that we take the design of (c) and replace the two servers by a single unit that runs twice as fast (i.e. it can process two minutes of work per unit of "wall clock" time). Suppose that this unit costs \$1000 per 8 hour day to operate versus \$400 per 8 hour day for each of the unit rate servers. We are charged \$1 per minute for each minute that a customer spends in the system. Should we replace the design of (c) with the design of (d)?

<u>**Solution:**</u> We compare the average cost to serve one customer since the arrival rate for both designs are the same. For design (c), the cost of interest is

$$\left(\frac{400}{8}\cdot 2 + 1\cdot 60\right)\cdot\frac{13}{40} = \$52$$

The average time-in-system for the design (d) is clearly

$$\frac{1}{20-10} = \frac{1}{10}(\text{hrs})$$

So the cost of interest for design (d) is

$$\left(\frac{1000}{8} + 1 \cdot 60\right) \cdot \frac{1}{10} = \$18.5$$

so we should replace the design (c) with the design (d).

(e) [6 PTS] Suppose that we return to the setting of (a). We have two service units, each serving customers at unit rate. The arrival process to each unit is Poisson. For the first unit, the arrival stream has an arrival rate of 8 customers per hour, and the service times are exponentially distributed with a mean of 12 minutes. The arrival stream to the second unit has an associated rate of 4 customers per hour, and the service times for these customers are exponentially distributed with a mean of 2.4 minutes. We decide to re-engineer the system by redirecting 50% of unit 1's customers to unit 2 instead. We do this by flipping a coin upon the arrival of each customer to unit 1. If the coin toss comes up heads, we send the customer to unit 2's waiting line instead; otherwise, the customer joins unit 1's line. What is the average time-in-system for a customer in this design?

Solution: We may view this system as an open Jackson network. Obviously, the total arrival rates for unit 1 and unit 2 are $\lambda_1 = 8/2 = 4$ and $\lambda_2 = 4 + 8/2 = 8$, respectively. The each unit, it's a M/M/1 queue. So we have the individual number-in-system for each unit

$$L_1 = \frac{\lambda_1}{\mu_1 - \lambda_1} = \frac{4}{5-4} = 4$$

and

$$L_2 = \frac{\lambda_2}{\mu_2 - \lambda_2} = \frac{8}{25-8} = \frac{8}{17}.$$

By Little's Law, the average time-in-system is given by

$$W = \frac{L_1 + L_2}{\lambda_1 + \lambda_2} = \frac{4 + \frac{8}{17}}{4+8} = \frac{19}{51}(\text{hrs}).$$

Question 2 (13 POINTS)

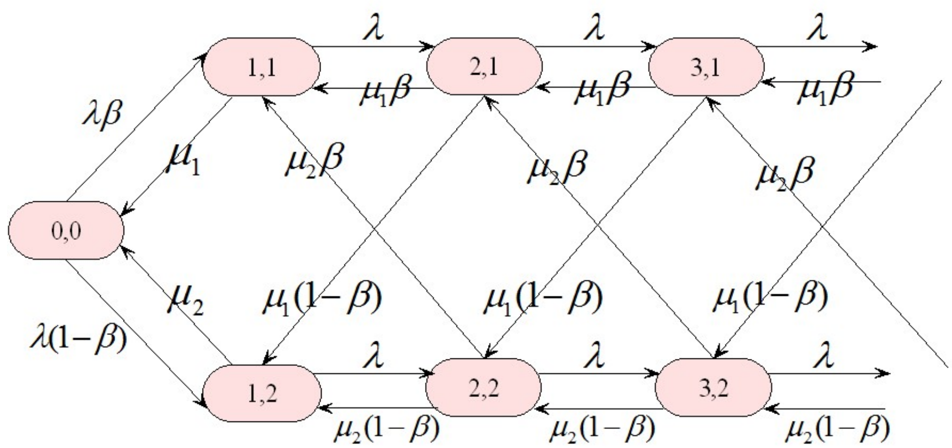
Consider a model of a production facility in which customers arrive according to a Poisson process having a rate 8 orders per 5 days (work) week. Each order requires a hyper-exponential processing time with probability density given by

$$f(x) = \frac{1}{3}e^{-x} + \frac{2}{3} \cdot (2e^{-2x})$$

(when measured in terms of days). Suppose that each order is canceled if it is not processed to completion after a random amount of time that is exponentially distributed with a mean of 10 (work) days.

(a) [4 PTS] Formulate the above problem as a continuous-time Markov chain model.

Solution: We can interpret the hyper-exponential service time as follows: with probability $\beta = 1/3$, the server runs as an exponential with rate $\mu_1 = 1$ and with probability $1-\beta = 2/3$, the server runs as an exponential with rate $\mu_2 = 2$. Let $\lambda = 8/5$ be the arrival rate. Let $X(t)$ be the number of customers in system at t and $J(t)$ be the type of customer in service. Let $J(t)$ be 0 if $X(t) = 0$. Then, $\{(X(t), J(t)), t \geq 0\}$ will be a CTMC with state space $\{(0,0), (n,j), n = 1,2,\ldots; j = 1,2\}$. The state transition diagram is shown as below.



(b) [3 PTS] Write down (but do not solve) the steady-state equations for this model.

$$\lambda\pi(0,0) = \mu_1\pi(1,1) + \mu_2\pi(1,2)$$
$$(\mu_1 + \lambda)\pi(1,1) = \lambda\beta\pi(0,0) + \mu_1\beta\pi(2,1) + \mu_2\beta\pi(2,2)$$
$$(\mu_2 + \lambda)\pi(1,2) = \lambda(1-\beta)\pi(0,0) + \mu_1(1-\beta)\pi(2,1) + \mu_2(1-\beta)\pi(2,2)$$
$$(\mu_1 + \lambda)\pi(n,1) = \lambda\pi(n-1,1) + \mu_1\beta\pi(n+1,1) + \mu_2\beta\pi(n+1,2), \quad n \geq 2$$
$$(\mu_2 + \lambda)\pi(n,2) = \lambda\pi(n-1,2) + \mu_1(1-\beta)\pi(n+1,1) + \mu_2(1-\beta)\pi(n+1,2), \quad n \geq 2$$

(c) [4 PTS] Provide an algorithm for generating random variables with the above hyper-exponential distribution (given a source of i.i.d. uniform (0,1) r.v.'s).

Solution: We use inverse transformation method. The CDF of this hyper-exponential distribution is

$$F(x) = \int_0^x \frac{1}{3}e^{-z} + \frac{2}{3} \cdot (2e^{-2z})\, dz = 1 - \frac{1}{3}e^{-x} - \frac{2}{3} \cdot (e^{-2x})$$

Then we set $F(X) = U$ where U is a uniform (0,1) r.v. and solve for U. Note that

$$1 - \frac{1}{3}e^{-X} - \frac{2}{3} \cdot (e^{-2X}) = U$$

can be seen as a quadratic function in e^{-X} and thus

$$e^{-X} = \frac{-1 + \sqrt{25 - 24U}}{4}$$

So

$$X = \log\left(\frac{4}{-1 + \sqrt{25 - 24U}}\right)$$

(d) [2 PTS] Suppose that we wish to compute the expected number of orders in the system in 5 days, given that the facility is currently idle. We do 4 simulations, obtaining the outcomes 2,3,2,5. Provide an approximate 95% confidence interval for the expected value.
$P(N(0,1) > 1.64) = 0.05, P(N(0,1) > 2.32) = 0.01, P(N(0,1) > 1.96) = 0.025$

Solution:

$$\bar{x}_4 = (2 + 3 + 2 + 5)/4 = 3$$

$$s^2 = \frac{1}{3}[(2-3)^2 + (3-3)^2 + (2-3)^2 + (5-3)^2] = 2$$

The 95% confidence interval is

$$(3 - 1.96 \cdot 2/\sqrt{2}, 3 + 1.96 \cdot 2/\sqrt{2}) \approx (1.61, 4.39)$$

Question 3 (11 points)

A newspaper vendor at San Francisco International Airport is considering her ordering policy for Sunday newspapers. The demand for the Sunday newspapers is uniformly distributed between 50 and 100. Each newspaper costs $1 to purchase from the distributor, and is sold for $3. For each newspaper that is left over at the end of the day, the airport charges a recycling fee of $0.5. To encourage vendors to not order excessively (and to reduce the garbage hauling costs for the airport), there is an additional flat fee of $10.00 charged whenever 10 or more copies need to be recycled.

(a) [4 pts] If the order quantity is y and the demand is D, write down an expression for $P(D, y)$, the profit corresponding to D and y.

Solution:

$$P(D, y) = 3\min(D, y) - y - 0.5(y - D)^{+} - 10I(y - D \geq 10)$$

(b) [4 PTS] Compute the expected value of $P(D, y)$. (You may leave the integrals without simplifying.)

Solution: For $y \in [60, 100]$,

$$E[P(D,y)] = 3\int_{50}^{y} \frac{x}{50}\,dx + 3\int_{y}^{100} \frac{y}{50}\,dx - y - \int_{50}^{y} \frac{y-x}{50}\,dx - 10\int_{50}^{y-10} \frac{1}{50}\,dx$$

For $y \in [50, 60)$,

$$E[P(D,y)] = 3\int_{50}^{y} \frac{x}{50}\,dx + 3\int_{y}^{100} \frac{y}{50}\,dx - y - \int_{50}^{y} \frac{y-x}{50}\,dx$$

Hence,

$$E[P(D,y)] = \begin{cases} -\frac{y^2}{25} + \frac{29y}{5} - 88 & 60 \le y \le 100 \\ -\frac{y^2}{25} + 6y - 100 & 50 \le y < 60 \end{cases}$$

(c) [3 PTS] Compute the optimal order quantity. (Set up the equations; you do not need to solve to a complete answer.)

Solution:

$$\frac{dE[P(D,y)]}{dy} = \begin{cases} -\frac{2y}{25} + \frac{29}{5} & 60 \leq y \leq 100 \\ -\frac{2y}{25} + 6 & 50 \leq y < 60 \end{cases}$$

Set $\frac{dE[P(D,y)]}{dy} = 0$ and solve for y. We get $y = 145/2$

Question 4 (13 POINTS)

Joe enjoys betting on basketball games. He bets on every game. If the team won (lost) the last game and he bet on the team to win (lose) that game, then he bets that the team will win (lose) the next game, and wagers \$2. If he loses his bet, Joe loses all the money wagered. When Joe bets that the team will win (lose) and loses his wager, he bets \$1 on the next game that the team will win (lose) with probability 1/4. When Joe wins a bet, he obtains a winnings equal to the amount wagered. The team's record of success/failure follows a Markov chain with the transition matrix:

	win	lose
win	3/4	1/4
lose	1/2	1/2

We wish to study the long-run rate at which Joe is winning money.

(a) [3 PTS] What is an appropriate state space for this problem?

Solution: We use the triplet (X, Y, Z) to denote the states, where $X \in \{W, L\}$, $Y \in \{C, I\}$, $Z \in \{1, 2\}$, i.e., X represents the team Joe bets on wins or loses, Y represents Joe is correct or incorrect, Z represents the wager. Then there are 8 different states:

$$(W, C, 1), (W, C, 2), (W, I, 1), (W, I, 2), (L, C, 1), (L, C, 2), (L, I, 1), (L, I, 2)$$

(b) [4 PTS] Write down the transition matrix for the Markov chain associated with part (a).

Solution:

	$(W,C,1)$	$(W,C,2)$	$(W,I,1)$	$(W,I,2)$	$(L,C,1)$	$(L,C,2)$	$(L,I,1)$	$(L,I,2)$
$(W,C,1)$	0	3/4	0	0	0	0	0	1/4
$(W,C,2)$	0	3/4	0	0	0	0	0	1/4
$(W,I,1)$	9/16	0	3/16	0	1/16	0	3/16	0
$(W,I,2)$	9/16	0	3/16	0	1/16	0	3/16	0
$(L,C,1)$	0	0	0	1/2	0	1/2	0	0
$(L,C,2)$	0	0	0	1/2	0	1/2	0	0
$(L,I,1)$	1/8	0	3/8	0	3/8	0	1/8	0
$(L,I,2)$	1/8	0	3/8	0	3/8	0	1/8	0

(c) [3 PTS] Write down the steady-state equations for the transition matrix postulated in part (b).

Solution: We use $1, 2, \ldots, 8$ to denote the 8 states $(W,C,1), (W,C,2), (W,I,1), (W,I,2), (L,C,1), (L,C,2), (L,I,1), (L,I,2)$, respectively. Then,

$$\begin{aligned}
\pi(1) &= \frac{9}{16}\pi(3) + \frac{9}{16}\pi(4) + \frac{1}{8}\pi(7) + \frac{1}{8}\pi(8) \\
\pi(2) &= \frac{3}{4}\pi(1) + \frac{3}{4}\pi(2) \\
\pi(3) &= \frac{3}{16}\pi(3) + \frac{3}{16}\pi(4) + \frac{3}{8}\pi(7) + \frac{3}{8}\pi(8) \\
\pi(4) &= \frac{1}{2}\pi(5) + \frac{1}{2}\pi(6) \\
\pi(5) &= \frac{1}{16}\pi(3) + \frac{1}{16}\pi(4) + \frac{3}{8}\pi(7) + \frac{3}{8}\pi(8) \\
\pi(6) &= \frac{1}{2}\pi(5) + \frac{1}{2}\pi(6) \\
\pi(7) &= \frac{3}{16}\pi(3) + \frac{3}{16}\pi(4) + \frac{1}{8}\pi(7) + \frac{1}{8}\pi(8) \\
\pi(8) &= \frac{1}{4}\pi(1) + \frac{1}{4}\pi(2) \\
1 &= \sum_{i=1}^{8} \pi(i)
\end{aligned}$$

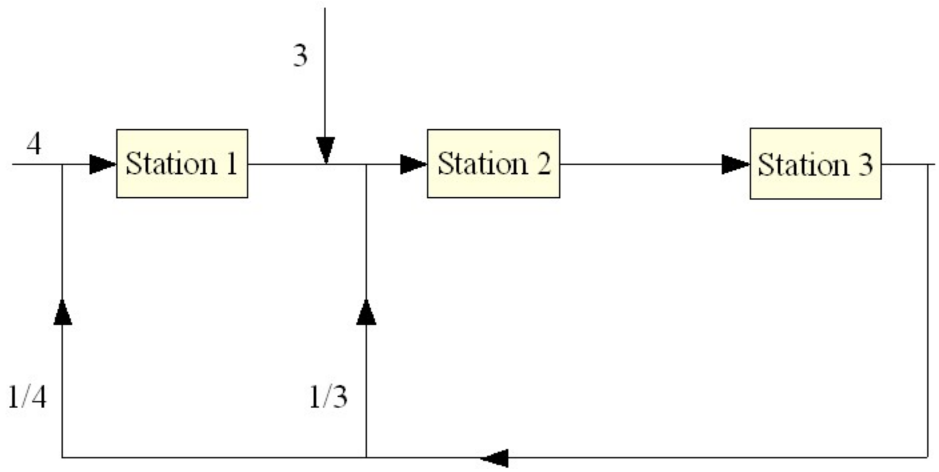
(d) [3 PTS] Suppose you have computed the steady-state probabilities. How would you compute an approximation to the amount won by Joe over 100 games?

Solution:

$$100[(W,C,1)+2(W,C,2)-(W,I,1)-2(W,I,2)+(L,C,1)+2(L,C,2)-(L,I,1)-2(L,I,2)]$$

Question 5 (12 POINTS)

A production facility consists of two processing stages, and an inspection station. The two processing workcenters and inspection station are denoted below as stations 1, 2, and 3 respectively:



The arrival rate to the first station is 4 jobs/hr., whereas the orders arrive directly to the second station at a rate of 3 jobs/hr. One quarter of the jobs passing through the inspection station need to be routed back to station 1 for additional processing; one third are routed to station 2 for additional processing. Station 2 and 3 process in an average of 3 minutes per job, whereas station 1 takes an average of 5 minutes of a job.

(a) [4 PTS] What are the effective arrival rates to the three stations?

Solution:

$$\begin{aligned}\gamma_1 &= 4 + \frac{1}{4}\gamma_3 \\ \gamma_2 &= 3 + \gamma_1 + \frac{1}{3}\gamma_3 \\ \gamma_3 &= \gamma_2\end{aligned}$$

So

$$\gamma_1 = \frac{41}{5}, \gamma_2 = \gamma_3 = \frac{84}{5}$$

(b) [4 PTS] What is the long-run fraction of time that the processing stations are both empty? Assume here (and throughout the reminder of the question) that the external arrival processes are Poisson and the processing times for the 3 stations are exponentially distributed.

Solution: All the three stations are M/M/1 queue. So the steady-state probabilities are

$$\begin{aligned}\pi_1(0) &= 1-\frac{\gamma_1}{\mu_1}=1-\frac{41/5}{12}=\frac{19}{60}\\ \pi_2(0) &= 1-\frac{\gamma_2}{\mu_2}=1-\frac{84/5}{20}=\frac{4}{25}\end{aligned}$$

By "PASTA", the long-run fraction of time that the processing stations are both empty is

$$\pi_1(0)\pi_2(0)=\frac{19}{375}$$

(c) [4 PTS] What is the average amount of time a job spends in the facility?

Solution: The queue lengths are

$$\begin{aligned}L_1 &= \frac{\gamma_1}{\mu_1-\gamma_1}=\frac{41/5}{12-41/5}=\frac{41}{19}\\ L_2 &= \frac{\gamma_2}{\mu_2-\gamma_2}=\frac{84/5}{20-84/5}=\frac{21}{4}\\ L_3 &= \frac{\gamma_3}{\mu_3-\gamma_3}=\frac{84/5}{20-84/5}=\frac{21}{4}\end{aligned}$$

By Little's law,

$$W=\frac{L_1+L_2+L_3}{\lambda}=\frac{\frac{41}{19}+\frac{21}{4}+\frac{21}{4}}{3+4}=\frac{481}{266}$$

where λ is the total external arrival rate.