Midterm Exam

Place all answers on the question sheet provided. The exam is open textbook (Hillier and Lieberman) and open notes/handouts/homework. Write all answers clearly and in complete sentences. All answers should be supported by analysis or an argument.

First Name: ________________________

Last Name: ________________________

1(a)	1(b)	1(c)	1(d)	2(a)	2(b)	2(c)	2(d)	3(a)	3 (b)	3(c)	3(d)	3(e)	3(f)

The Stanford Honor Code

1. The Honor Code is an undertaking of the students, individually and collectively:
   (a) That they will not give or receive aid in examinations; that they will not give or receive unpermitted aid in class work, in preparation of reports, or in any other work that is to be used by the instructor as the basis of grading;
   (b) that they will do their share and take an active part in seeing to it that others as well as themselves uphold the spirit and letter of the Honor Code.
2. The faculty on its part manifests its confidence in the honor of its students by refraining from proctoring examinations and from taking unusual and unreasonable precautions to prevent the forms of dishonesty mentioned above. The faculty will also avoid, as far as practicable, academic procedures that create temptations to violate the Honor Code.
3. While the faculty alone has the right and obligation to set academic requirements, the students and faculty will work together to establish optimal conditions for honorable academic work. I acknowledge and accept the Honor Code.

Signature: ________________________

Question 1 (13 POINTS)

(a) [4 PTS] Give an algorithm for generating a random variable X having cumulative distribution function

$$F_X(x) = \begin{cases} 0, & x \le 0 \\ 10x^3 - 15x^4 + 6x^5, & 0 \le x \le 1 \\ 1, & x \ge 1. \end{cases}$$

Solution We use the acceptance–rejection algorithm. Since the density function $f_X(x)$ of X is given by

$$f_X(x) = F'_X(x) = \begin{cases} 30x^2(1-x)^2 & , 0 \le x \le 1 \\ 0 & , \text{otherwise} \end{cases}$$

and $f'_X(x)$ is given by

$$f'_X(x) = \begin{cases} 60x(1-x)(1-2x) & , 0 \le x \le 1 \\ 0 & , \text{otherwise} \end{cases}$$

$M = \max_{0 \le x \le 1} f(x) = f(1/2) = 15/8$. The acceptance–rejection algorithm is as follows:

i) Generate U_1, U_2 iid $\sim$ Unif[0, 1].

ii) If $MU_2 \le f(U_1) = 30U_1^2(1-U_1)^2$, then return $X = U_1$, otherwise return to step i).

(b) [1 PTS] How many uniform random numbers does it take, on average, to generate a single X (using your algorithm)?

Solution In each iteration of the acceptance–rejection algorithm, the probability of acceptance is $p = 1/M = 8/15$. Since it takes two uniform random variables per iteration, the expected number of uniform random variables until one generates a single X is $2(1 \cdot p + 2 \cdot (1-p)p + 3 \cdot (1-p)^2 p + \cdots) = 2/p = 15/4$.

(c) [5 PTS] Suppose we wish to model an order fulfillment system in which both high priority and low priority orders are served. Within each priority class, orders are served in the order in which they arrive. Any order on which processing initiated is processed until completion, regardless of priority level. If both high priority and low priority orders are present in the "order queue", a high priority order is always selected as the next order to be processed.

Low priority order arrive at times 3, 5, 9, 12, and 17, with associated order processing times of 4, 5, 1, 3, and 8. High order priority orders arrive at time 4, 11, 16, 23, and 25, with order processing times of 3 time units each. The facility processes orders at unit rate. How many low priority orders have been completed by time 17.5?

Solution Let X_t be the number of customers in-system at time t. At any point in time, there are two possible events that can happen next; departure or arrival. So, we compare the next arrival time to the next departure time and decide which one happens first. Once this is decided, we move on to the time when the next event happens and update the system accordingly. The following table contains all the information that we need to keep track of the system at any given time. ✓ indicates the event that happens next. L_i and H_i indicate the ith low priority and high priority orders, respectively. Boldfaced order indicates the order being served by the server.

time	next arrival time	next departure time	in the system
3	4 ✓	7	$L_1 \mid$
4	5 ✓	7	$L_1 \mid H_1$
5	9	7 ✓	$L_1 \mid H_1 L_2$
7	9 ✓	10	$H_1 \mid L_2$
9	11	10 ✓	$H_1 \mid L_2 L_3$
10	11 ✓	15	$L_2 \mid L_3$
11	12 ✓	15	$L_2 \mid L_3 H_2$
12	16	15 ✓	$L_2 \mid L_3 H_2 L_4$
15	16 ✓	18	$H_2 \mid L_3 L_4$
16	17 ✓	18	$H_2 \mid L_3 L_4 H_3$
17	23	18 ✓	$H_2 \mid L_3 L_4 H_3 L_5$

Hence two low priority orders have been completed by time 17.5.

(d) [3 PTS] You simulate the above model four times, and find that the numbers of low priority customers completed by time 17.5 are 2, 5, 1, and 3, respectively. Produce a 95% confidence interval for the expected number of low priority orders that will be completed by time 17.5. Note : $P(N(0,1) \geq 1.64) = 0.05, P(N(0,1) \leq -1.96) = 0.025, P(N(0,1) \geq 2.58) = 0.005$.

Solution The 95% confidence interval is given by

$$\left[\overline{X}_4 - z\sqrt{\frac{V_4}{4}}, \overline{X}_4 + z\sqrt{\frac{V_4}{4}}\right]$$

where

$$\begin{aligned}
\overline{X}_4 &= \frac{2+5+1+3}{4} = \frac{11}{4}, \\
\overline{V}_4 &= \frac{1}{3}\left\{\left(2-\frac{11}{4}\right)^2 + \left(5-\frac{11}{4}\right)^2 + \left(1-\frac{11}{4}\right)^2 + \left(3-\frac{11}{4}\right)^2\right\}, \\
z &= \Phi^{-1}(0.975) = 1.96.
\end{aligned}$$

Question 2 (12 POINTS)

The population of deer on an island is observed four years in a row (2004 through 2007), with observed levels of 7, 10, 19, and 35 deer respectively. You believe that the population is expanding at an exponential rate, so that if Y_n is the number of deer in year n, $E(\log(Y_n)) = a^*n + b^*$.

(a) [4 PTS] Compute your best estimates of a^* and b^*.

$$\frac{1}{4}\sum_{i=1}^{4} Y_i = 17.75 \qquad \frac{1}{4}\sum_{i=1}^{4} Y_i^2 = 433.75$$
$$\frac{1}{4}\sum_{i=1}^{4} \log(Y_i) = 2.69 \qquad \frac{1}{4}\sum_{i=1}^{4} (\log(Y_i))^2 = 7.60$$
$$\frac{1}{4}\sum_{i=1}^{4} t_i = 2005.5 \qquad \sum_{i=1}^{4} t_i^2 = 16088126$$
$$\sum_{i=1}^{4} \log(t_i) = 30.4 \qquad \sum_{i=1}^{4} t_iY_i = 142437$$
$$\sum_{i=1}^{4} t_i \log(Y_i) = 21558.$$

Solution Let $t_1 = 2004, t_2 = 2005, t_3 = 2006, t_4 = 2007$ and $Y_1 = 7, Y_2 = 10, Y_3 = 19, Y_4 = 35$.

$$\begin{aligned}
\widehat{a} &= \frac{\sum_{i=1}^4 t_i \log(Y_i) - 4\left(\sum_{i=1}^4 \log Y_i/4\right)\left(\sum_{i=1}^4 t_i/4\right)}{\sum_{i=1}^4 t_i^2 - 4\left(\sum_{i=1}^4 t_i/4\right)\left(\sum_{i=1}^4 t_i/4\right)} \\
&= \frac{21558 - 4(2.69)(2005.5)}{16088126 - 4(2005.5)^2} = -4.236
\end{aligned}$$

(this number appears to be negative due to rounded errors of above averages/sums)

$$\begin{aligned}
\widehat{b} &= \frac{1}{4}\sum_{i=1}^{4} \log Y_i - \widehat{a}\frac{1}{4}\sum_{i=1}^{4} t_i \\
&= 2.69 - (-4.236)(2005.5) = 8498
\end{aligned}$$

(b) [4 PTS] Based on the model in (a), compute the best prediction of the number of deer on the island in 2009.

Solution

$$\begin{aligned} \log Y_{2009} &= \widehat{a} \cdot 2009 + \widehat{b} \\ &= -4.236 \cdot 2009 + 8498 = -12.124 \\ \therefore \quad Y_{2009} &= e^{-12.124} = 5.43 \times 10^{-6} \end{aligned}$$

(c) [2 PTS] Suppose that on a second island the deer population has stabilized, and that the deer population in the years 2004 through 2007 is 12, 15, 10, and, 12, respectively. You estimate the parameters for a second–order autoregressive model as

$$Y_{n+1} = 0.23Y_n + 1.05Y_{n-1} - 6 + \varepsilon_{n+1}$$

where the ε_n's are iid standard normal rv's. Use this model to compute your best prediction of the deer population in 2008.

Solution

$$\begin{aligned} Y_{2008} &= 0.23Y_{2007} + 1.05Y_{2006} - 6 \\ &= 0.23(12) + 1.05(10) - 6 \\ &= 7.26. \end{aligned}$$

(d) [2 PTS] Now use the model to compute your best prediction of the deer population in 2009.

Solution

$$\begin{aligned} Y_{2009} &= 0.23Y_{2008} + 1.05Y_{2007} - 6 \\ &= 0.23(7.26) + 1.05(12) - 6 \\ &= 8.27. \end{aligned}$$

Question 3 (15 POINTS)

Two players, A and B, are playing cards. If player A wins two games in a row, then he wins the next game with probability 0.7. If player B wins two games in a row, she wins the next game with probability 0.6. If one of the previous two games was won by A, and one was won by B, then the players have equal chances to win the next game.

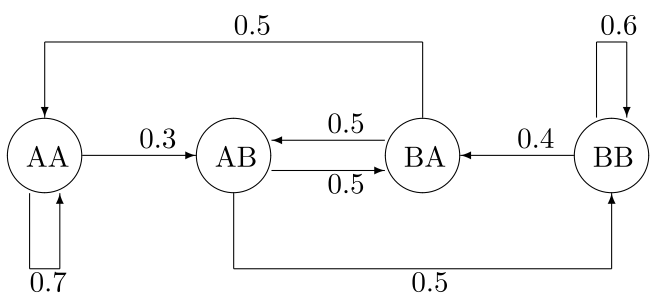
(a) [1 PTS] Model the system as a Markov Chain. Define the state space.

Solution Consider a Markov chain $(X_n : n = 0, 1, \ldots)$ with states:

AA	if A wins the current game and A has won the previous game,
AB	if A wins the current game and B has won the previous game,
BA	if B wins the current game and A has won the previous game,
BB	if B wins the current game and B has won the previous game.

(b) [1 PTS] Draw the transition diagram and label the transition probabilities.

Solution



(c) [4 PTS] Write down the transition matrix. Is your Markov Chain irreducible or not? Aperiodic or periodic?

Solution The transition matrix is

$$P = \begin{bmatrix} 0.7 & 0.3 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0.4 & 0.6 \end{bmatrix}.$$

This Markov chain is ireducible because from any state there is a path with positive probability to other states (including itself). Because this Markov chain is irreducible and state AA has a self–loop, this chain is aperiodic.

(d) [3 PTS] If the player A has had two victories in a row, what is the probability that he will win the next two games?

Solution

$$\begin{aligned} & P(X_{n+2} = AA \text{ and } X_{n+1} = AA | X_n = AA) \\ = \; & P(X_{n+2} = AA | X_{n+1} = AA) \cdot P(X_{n+1} = AA | X_n = AA) \\ = \; & P(1,1) \cdot P(1,1) = 0.7^2 = 0.49. \end{aligned}$$

(e) [4 PTS] What is the (approximate) probability that in the 1000th game, Player A has just won and has won a series of at least three victories in a row (including the current victory on the 1000th game)?

Solution

$$\begin{aligned}
& P(X_{999} = AA \text{ and } X_{1000} = AA) \\
= \; & P(X_{999} = AA) \cdot P(X_{1000} = AA | X_{999} = AA) \\
\approx \; & \pi(AA) \cdot 0.7
\end{aligned}$$

where $\pi = [\pi(AA)\ \pi(AB)\ \pi(BA)\ \pi(BB)]$ satisfies

$$\begin{aligned}
0.7\pi(AA) + 0.5\pi(BA) &= \pi(AA) \\
0.3\pi(AA) + 0.5\pi(BA) &= \pi(AB) \\
0.5\pi(AB) + 0.4\pi(BB) &= \pi(BA) \\
0.5\pi(AB) + 0.6\pi(BB) &= \pi(BB) \\
\pi(AA) + \pi(AB) + \pi(BA) + \pi(BB) &= 1.
\end{aligned}$$

Hence, $\pi = [0.3390\ 0.2034\ 0.2034\ 0.2542]$ and $P(X_{999} = AA \text{ and } X_{1000} = AA) = 0.339 \cdot 0.7 = 0.2373$.

(f) [2 PTS] If the coach of Player B pays Player B \$5 every time that she wins and has also won the preceding game, how much will the coach pay out (approximately) over the first 100 games? (Hint: Note that the pay–out over 100 games should be approximately 100 times the long–run average pay–out per game.)

Solution

$$\begin{aligned} & 100 \times (\text{ long–run average pay–out per game }) \\ = & \ \$100 \times (5 \cdot \pi(BB)) \\ = & \ \$100 \cdot 5 \cdot 0.2542 \\ = & \ \$127.1 \end{aligned}$$