Final Exam Solutions

Place all answers on the question sheet provided. The exam is open textbook (Hillier and Lieberman) and open notes/handouts/homework. Write all answers clearly and in complete sentences. All answers should be supported by analysis or an argument.

First Name: ____________________

Last Name: ____________________

1(a)	1(b)	1(c)	1(d)	1(e)	1(f)	2(a)	2(b)	2(c)	2(d)

2(e)	2(f)	2(g)	3(a)	3(b)	4(a)	4(b)	4(c)	4(d)	4(e)	Total

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(1) That they will not give or receive aid in examinations; that they will not give or receive unpermitted aid in class work, in preparation of reports, or in any other work that is to be used by the instructor as the basis of grading;

(2) That they will do their share and take an active part in seeing to it that others as well as themselves uphold the spirit and letter of the Honor Code.

B. The faculty on its part manifests its confidence in the honor of its students by refraining from proctoring examinations and from taking unusual and unreasonable precautions to prevent the forms of dishonesty mentioned above. The faculty will also avoid, as far as practicable, academic procedures that create temptations to violate the Honor Code.

C. While the faculty alone has the right and obligation to set academic requirements, the students and faculty will work together to establish optimal conditions for honorable academic work.

I acknowledge and accept the Honor Code.

Signature: ____________________

Question 1

(24 POINTS) A car dealership currently has 15 cars on its lot from the 2006–07 model year. The new 2007–08 vehicles will be arriving on the lot in the next month. The vehicle manufacturer is providing 2006–07 vehicles to its dealers at a cost of $\$20K$ (the final vehicles from its 2006–07 production run); the dealer cost for 2007–08 vehicles will be $\$25K$ per vehicle. The dealer believes that 2006–07 vehicles can be sold for $\$24K$ per vehicle, while 2007–08 vehicles can be sold for $\$32K$ per vehicle. Any unsold 2006–07 vehicles will be sold to a local car rental company for $\$16K$ per vehicle. The dealer believes the demand D for 2006–07 vehicles will be a uniformly distributed continuous random variable over the interval [10, 50]. In addition, the dealer believes that 20% of the 2006–07 buyers can be induced to buy 2007–08 vehicles if the dealer is out of 2006–07 cars.

(a) [4 PTS] Let y be the number of 2006–07 vehicles acquired by the dealer (including those vehicles currently on the lot). Write down $P(D, y)$ = the profit associated with demand D and vehicle quantity y.

Solution : Assume $y \geq 15$.

$$P_1(D, y) = 24\min(D, y) + 0.2 \cdot 7 \cdot (D - y)^+ + 16(y - D)^+ - 20(y - 15).$$

(b) [4 PTS] Compute the expected profit at vehicle order quantity y.

Solution :

$$\begin{aligned} EP_1(D, y) &= 24\Big\{\int_{10}^{y} x\frac{1}{40}dx + \int_{y}^{50} y \cdot \frac{1}{40}dx\Big\} \\ &\quad +1.4\int_{y}^{50}(x - y)\frac{1}{40}dx + 16\int_{10}^{y}(y - x)\frac{1}{40}dx - 20(y - 15) \\ &= -0.0285y^2 + 4.25y + 333.75. \end{aligned}$$

(c) [4 PTS] Compute the optimal size of the order for 2006–07 vehicles.

Solution :

$$\frac{d}{dy}EP_1(D,y)|_{y=y^*} = -0.165y^* + 4.25 = 0$$

Hence $y^* = 25.7576$. Since $EP_1(D,25) = 388.44 < EP_1(D,26) = 388.48$, 26 is the optimal inventory level and the optimal size of the order for 2006–07 vehicle is 26-15 = 11.

(d) [4 PTS] Suppose that the manufacturer provides an additional dealer incentive of $\$50K$ if the dealership sells more than 40 2006–07 vehicles. Write down the new $P(D,y)$.

Solution : $P_2(D,y) = P_1(D,y) + 50I(\min(D,y) \geq 40)$

(e) [4 PTS] Compute the new expected profit at vehicle order quantity y.

Solution :

$$\begin{aligned} EP_2(D,y) &= \begin{cases} EP_1(D,y) + 50P(D \geq 40), & \text{if } y \geq 40, \\ EP_1(D,y), & \text{otherwise} \end{cases} \\ &= \begin{cases} EP_1(D,y) + 12.5, & \text{if } y \geq 40 \\ EP_1(D,y), & \text{otherwise} \end{cases} \end{aligned}$$

(f) [4 PTS] Compute the new optimal size of the order for 2006–07 vehicles.

Solution : Compare $EP_1(D,25) = 388.4375$ and $EP_1(D,40) + 12.5 = 384.25$. Since $EP_1(D,25) > EP_1(D,40) + 12.5$, the optimal size of the order for 2006–07 vehicle is 26-15 = 11.

Question 2

(26 POINTS) Suppose that a company rents warehouse space as a means of distributing its product. The company observes demand over a period of one week. Starting at $t = 0$, orders (for one item each) occur at $t = 1.3, 2.0, 4.8,$ and 5.1 over the seven days (time measured in days).

(a) [4 PTS] Suppose that the arrival rate for orders is λ. What is the likelihood of observing the above demand when we assume that orders follow a Poisson process?

Solution :

$$\begin{aligned} L(\lambda) &= \lambda e^{-1.3\lambda}\lambda e^{-0.7\lambda}\lambda e^{-2.8\lambda}\lambda e^{-0.3\lambda}e^{-1.9\lambda} \\ &= \lambda^4 e^{-7\lambda} \end{aligned}$$

(b) [2 PTS] What is your maximum likelihood estimator for λ?

Solution :

$$\begin{aligned} \frac{d}{d\lambda}\log L(\lambda)|_{\lambda=\lambda^*} &= \frac{d}{d\lambda}(4\log\lambda - 7\lambda)|_{\lambda=\lambda^*} \\ &= \frac{4}{\lambda^*} - 7 = 0. \end{aligned}$$

Hence $\lambda^* = \frac{4}{7}$.

(c) [4 PTS] Suppose that the company has decided to implement an (s, S) inventory policy with $s = 0$ and $S = 3$. Whenever the warehouse inventory level hits 0, an order for 3 additional items is placed. The delivery of the shipment takes an average of one day and follows an exponential distribution. Any orders that arrive when the warehouse is out of stock are lost. Write down the rate matrix of the continuous–time Markov chain you would use to analysis this problem.

Solution : Define states by inventory level {0, 1, 2, 3}. Then the rate matrix is

$$\begin{bmatrix} -1 & 0 & 0 & 1 \\ \lambda^* & -\lambda^* & 0 & 0 \\ 0 & \lambda^* & -\lambda^* & 0 \\ 0 & 0 & \lambda^* & -\lambda^* \end{bmatrix}$$

(d) [4 PTS] What is the long–run fraction of time that there is no inventory on hand?

Solution : Steady–state probabilities satisfy

$$\begin{aligned} \pi(0) &= \lambda^*\pi(1) \\ \lambda^*\pi(1) &= \lambda^*\pi(2) \\ \lambda^*\pi(2) &= \lambda^*\pi(3) \\ \pi(0) &= \lambda^*\pi(3) \\ \pi(0)+\pi(1)+\pi(2)+\pi(3) &= 1. \end{aligned}$$

Hence the long–run fraction of time that there is no inventory on hand is $\pi(0) = \frac{\lambda^*}{\lambda^*+3} = \frac{4}{25}$.

(e) [4 PTS] Suppose that the inventory charges \$20 per day per item to hold the inventory of the company. Lost sales cost the company \$100 in foregone profit. What is the expected total cost of running this inventory system over the next 100 days? (Include both inventory holding costs and foregone profit.)

Solution : $100(20\pi(1) + 40\pi(2) + 60\pi(3) + 100\lambda^*\pi(0)) = 4960.$

(f) [4 PTS] Suppose that the cost per delivery is \$50. Repeat part (e) including the previous costs. (Hint : Think about the expected steady–state jump rate out of state 0.)

Solution : $4960 + 100 \cdot 50 \cdot 1 \cdot \pi(0) = 7817.1.$

(g) [4 PTS] There is currently one item in the warehouse. What is the probability that all three of the next three orders will be fulfilled?

Solution : P(Ordered placed after the first demand arrives before the second demand) $= P(exp(1) < exp(\lambda)) = \frac{1}{1+\lambda^*} = \frac{7}{11}$.

Question 3

(7 POINTS) In this problem, we will "price" three different financial options. The basic pricing model is one in which the change in price on the underlying asset is $+\$1, \$0, -\$1$ with respective probabilities $1/4, 1/2$, and $1/4$ (and independent from day to day). The current price of the asset is \$2. Let P_i be the price of the asset i days from now.

(a) [3 PTS] Consider an option that expires in 2 days from now. The option pays out $[P_2 - 3]^+$ dollars. Compute the expected return from holding this option. (This is a "European option".)

day 0	day 1	day 2	payoff
		4	1
	3	3	0
2	2	2	0
	1	1	0
		0	0

Hence the expected return $= \frac{1}{4} \cdot \frac{1}{4} \cdot 1 = \frac{1}{16}$.

(b) [4 PTS] Consider an option that expires in 2 days from now. The option can be exercised on day 0 (the current day), day 1, or day 2. If the option is exercised on day i, the option returns $[P_i - 2]^+$ dollars. Compute the expected return from holding this option. (This is an "American option".)

Solution : Let $V(x, i)$ be the best expected return given that the price of the option is $\$x$ on day i and we haven't exercised the option yet. Then

$$
\begin{aligned}
V(4,2) &= \max\{4-2, 0\} = 2 \\
V(3,2) &= \max\{3-2, 0\} = 1 \\
V(2,2) &= \max\{2-2, 0\} = 0 \\
V(1,2) &= \max\{1-2, 0\} = 0 \\
V(0,2) &= \max\{0-2, 0\} = 0 \\
V(3,1) &= \max\{3-2, \frac{1}{4}V(4,2) + \frac{1}{2}V(3,2) + \frac{1}{4}V(2,2)\} = 1 \\
V(2,1) &= \max\{2-2, \frac{1}{4}V(3,2) + \frac{1}{2}V(2,2) + \frac{1}{4}V(1,2)\} = \frac{1}{4} \\
V(1,1) &= \max\{1-2, \frac{1}{4}V(2,2) + \frac{1}{2}V(1,2) + \frac{1}{4}V(0,2)\} = 0
\end{aligned}
$$

Hence the expected return from holding this option is $\frac{1}{4}V(3,1) + \frac{1}{2}V(2,1) + \frac{1}{4}V(1,1) = \frac{3}{8}$.

Question 4

(18 POINTS) Consider a production facility to which incoming order arrive at a rate of 7 per day. In the current facility, the two processing steps are arranged in tandem. Orders first queue up at machine 1 (a single machine with an infinite capacity waiting room), are processed there, are released to machine 2 (a single machine with an infinite capacity waiting room), are processed there, and then sent to the customers. The processing times at the two stations are exponentially distributed with means of 1/8 day and 1/10 day, respectively. We assume the arrival process is Poisson.

(a) [4 PTS] Model this as a network of queues model by formulating and solving the traffic equations.

Solution : Let γ_1 and γ_2 be total arrival rates to machine 1 and machine 2, respectively. Then $\gamma_1 = 7$ per day and $\gamma_2 = \gamma_1 = 7$ per day. The service rates μ_1 and μ_2 are 8 per day and 10 per day, respectively.

(b) [2 PTS] What is the long–run probabilities that the system is empty?

Solution :

$$\begin{aligned} & P(0 \text{ at machine 1 and 0 at machine 2}) \\ = \; & P(0 \text{ at machine 1})P(0 \text{ at machine 2}) \\ = \; & \left(1 - \frac{\gamma_1}{\mu_1}\right)\left(1 - \frac{\gamma_1}{\mu_2}\right) = \left(1 - \frac{7}{8}\right)\left(1 - \frac{7}{10}\right) = \frac{3}{80}. \end{aligned}$$

(c) [4 PTS] What is the steady–state average time that an order will spend in the system?

Solution : The average number of customers in the network, L, is $\frac{\gamma_1}{\mu_1-\gamma_1} + \frac{\gamma_2}{\mu_2-\gamma_2} = \frac{28}{3}$. By Little's law, $W = L/\lambda = 4/3$.

The management believes that the operation could be significantly improved by buying a new machine that can perform both processing steps. The processing time for each order on the new machine can be viewed as the sum of two exponential random variables, one having mean 1/16 day and the other having mean 1/20 day. Orders now queue up just once (in front of the new machine).

(d) [4 PTS] What is the steady–state average time that an order spends in the new system?

Solution : This is M/G/1 queue with the arrival rate $\lambda = 7$ and the service rate $\mu = \frac{1}{1/16+1/20} = 80/9$. Hence

$$\begin{aligned}
\rho &= \lambda/\mu = 63/80, \\
\sigma &= \left(\frac{1}{16}\right)^2 + \left(\frac{1}{20}\right)^2 = \frac{41}{6400} \\
W &= \frac{\lambda^2\sigma^2 + \rho}{2(1-\rho)\lambda} + \frac{1}{\mu} = 0.3779.
\end{aligned}$$

(e) [4 PTS] What is the approximate probability that an order spends more than 1 days in the system?

Solution : Since $\rho \approx 1$, by heavy traffic approximation,

$$\mathcal{W}_q \overset{D}{\approx} \frac{(\sigma_A^2 + \sigma_S^2)}{2(m_A - m_S)} exp(1)$$

where

$$\begin{aligned}
\sigma_A^2 &= \text{variance of inter-arrival time} = \frac{1}{49} \\
\sigma_S^2 &= \text{variance of service time} = \frac{41}{6400} \\
m_A &= \text{mean inter-arrival time} = \frac{1}{7} \\
m_S &= \text{mean service time} = \frac{9}{80} \\
exp(1) &= \text{exponential r.v. having mean 1}
\end{aligned}$$

Let S be the service time, then $S = exp(16) + exp(20)$.

$$\begin{aligned}
P(W > 1) &\approx P(\frac{\sigma_A^2 + \sigma_S^2}{2(m_A - m_S)} exp(1) + exp(16) + exp(20) > 1) \\
&= P(0.4416 exp(1) + exp(16) + exp(20) > 1) \\
&= \int_0^1 \int_0^{1-t} P(0.4413 exp(1) + s + t > 1 | exp(16) = s, exp(20) = t) 16e^{-16s} 20 exp^{-20t} dsdt \\
&\quad + \int_0^1 \int_{1-t}^{\infty} 16e^{-16s} 20e^{-20t} dsdt + \int_1^{\infty} 20e^{-20t} dt \\
&= \int_0^1 \int_0^{1-t} e^{-\frac{1-(s+t)}{0.4416}} 16e^{-16s} 20e^{-20t} dsdt \\
&\quad + \int_0^1 \int_{1-t}^{\infty} 16e^{-16s} 20e^{-20t} dsdt + \int_1^{\infty} 20e^{-20t} dt = 0.1365
\end{aligned}$$