

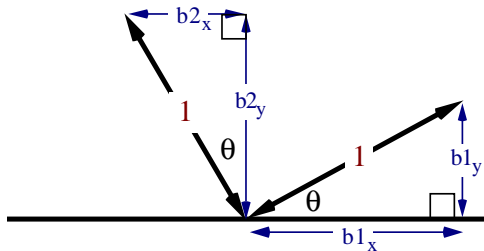
1.1 History of mathematics

Connect the mathematical invention with its approximate date of discovery.

Date	Mathematical invention
2000 B.C.	Invention of vectors
800 A.D.	Babylonian number system (based on 60)
1700 A.D.	Invention of the number 0
1900 A.D.	First widespread use of negative numbers

1.2 Sine and cosine review

Trigonometry plays a central role in kinematics, particularly in the formation of *rotation matrices*. Referring to the figure below, express b_{1x} , b_{1y} , b_{2x} , and b_{2y} in terms of $\sin(\theta)$ and $\cos(\theta)$.



$$b_{1x} = \text{[]} \quad b_{1y} = \text{[]}$$

$$b_{2x} = \text{[]} \quad b_{2y} = \text{[]}$$

1.3 Addition formula for cosine and its relationship to $\sin(x)^2$ and $\cos(x)^2$

Using the addition formula for cosine in equation (2.8), show

$$\sin(x)^2 = \frac{1 - \cos(2x)}{2} \qquad \cos(x)^2 = \frac{1 + \cos(2x)}{2}$$

1.4 Amplitude-phase examples

Put the following functions into *amplitude/phase form* by filling in the blanks.

$$-\sin(2t) = \sin(2t + \text{[]}) = \cos(2t + \text{[]})$$

$$-\cos(2t) = \sin(2t + \text{[]}) = \cos(2t + \text{[]})$$

$$-3*\sin(2t) + 4*\cos(2t) = 5*\sin(2t + \text{[]})$$

$$-3*\sin(2t + \frac{\pi}{3}) + 4*\cos(2t) = \text{[]}*\cos(2t + 0.82)$$

1.5 Manipulating sine and cosine functions into amplitude-phase form

An engineer is designing a wave-pool for a water park. The waves are created by two wave generators, and the amplitude, frequency, and phase of each wave at the center of the pool is measured separately and determined as

$$\begin{aligned} \text{wave}_A(t) &= 4 * \sin(3*t + \pi/4) \\ \text{wave}_B(t) &= 5 * \cos(3*t + \pi/3) \end{aligned}$$

With both wave generators operating, the resultant wave height at the center of the pool, is assumed to be $y(t) = \text{wave}_A(t) + \text{wave}_B(t)$. *First* using intuition or guessing and *then* using mathematics, determine $y(t)$ when it is expressed in terms of the amplitude¹ C , frequency ω , and phase ϕ as $y(t) = C * \sin(\omega*t + \phi)$. Comment on the accuracy of your intuition.

Result:

Intuition:	$y(t) = \text{ } * \sin(\text{ } * t + \text{ })$
Mathematics:	$y(t) = \text{ } * \sin(\text{ } * t + \text{ })$
Accuracy in predicting wave amplitude:	good/bad
Accuracy in predicting wave phase:	good/bad

1.6 Good and bad product rules for differentiation.

The **product rule for differentiation** that works when u and v are scalars, vectors, or matrices, is

$$\frac{d(u * v)}{dt} = \frac{du}{dt} * v + u * \frac{dv}{dt} \qquad \frac{d(u * v)}{dt} = u * \frac{dv}{dt} + v * \frac{du}{dt} \qquad \frac{d(u * v)}{dt} = v * \frac{du}{dt} + u * \frac{dv}{dt}$$

1.7 Review of differentiation and the product rule for differentiation.

Knowing x and y are variables that depend on the independent variable t (time), determine the time-derivative of the function f when²

$$f(t) = \sin(t) * \cos(x + y) * (\dot{x})^2 * e^t * \ln(y) / x$$

Result:

$$\begin{aligned} \frac{df}{dt} &= \cos(t) * \cos(x + y) * (\dot{x})^2 * e^t * \ln(y) / x \\ &- \text{ } \\ &+ \text{ } \\ &+ \text{ } \\ &+ \text{ } \\ &- \sin(t) * \cos(x + y) * (\dot{x})^3 * e^t * \ln(y) / x^2 \end{aligned}$$

¹In amplitude/phase form, the amplitude is non-negative.

²The symbol \dot{x} is an abbreviation for $\frac{dx}{dt}$, the time-derivative of x . Similarly, the symbol \ddot{x} is an abbreviation for $\frac{d^2x}{dt^2}$.

1.8 Differentiation of hyperbolic cosine and sine functions

The *hyperbolic cosine* and *hyperbolic sine* functions are defined below and plotted to the right.

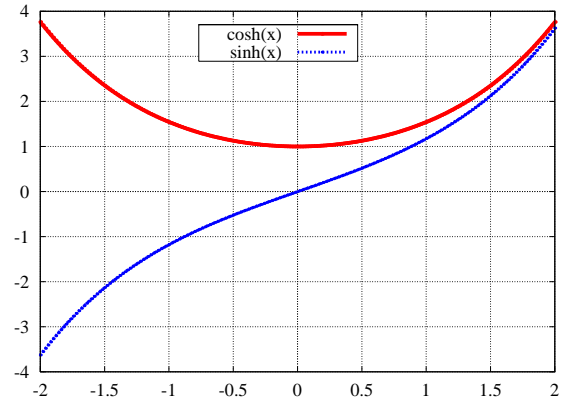
$$\cosh(x) \triangleq \frac{e^x + e^{-x}}{2} \quad \sinh(x) \triangleq \frac{e^x - e^{-x}}{2}$$

Show *how* to find their ordinary derivatives with respect to x in terms of hyperbolic functions.

Result:

$$\frac{d[\cosh(x)]}{dx} = \text{[Yellow Box]}$$

$$\frac{d[\sinh(x)]}{dx} = \text{[Yellow Box]}$$



1.9 Differentiation concepts

Calculate the ordinary time-derivative of $y = 5^t$.

Result:

$$\frac{dy}{dt} = \text{[Yellow Box]}$$

1.10 Differentiation concepts

A continuous function $z(t)$ depends on $x(t)$ and $y(t)$ (which also depend on time t) as

$$z = x + y^2 \sin(t)$$

At a certain instant of time, $y = 1$ and z simplifies to

$$z = x + \sin(t)$$

Find the time-derivative of z at the instant when $y = 1$.

Result:

$$\left. \frac{dz}{dt} \right|_{y=1} = \text{[Yellow Box]}$$

1.11 System identification for a 1st-order dynamic system (Coulomb friction)

The figure to the right shows a freely-spinning (non-driven) motor whose laboratory data for the spin rate ω correlates with the mathematical function

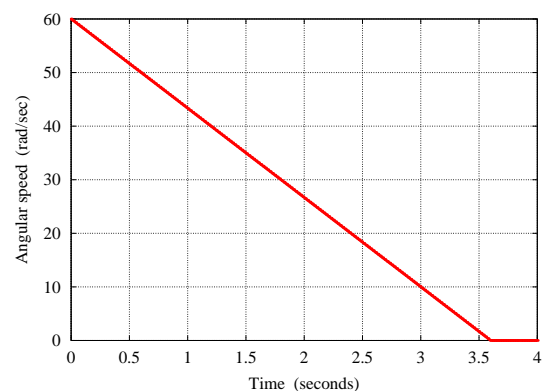
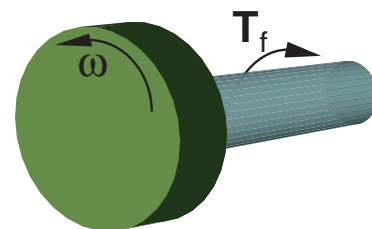
$$\omega(t) = \omega_0 + slope * t$$

where ω_0 is the initial spin rate and *slope* is the slope of the line. This mathematical function suggests a 1st-order ODE of the form^a

$$I \frac{d\omega}{dt} = -T_f$$

where I is a constant (the moment of inertia of the motor and its rotor and attachments about the motor's axis) and T_f is a constant representing the torque exerted by Coulomb friction.

^aAn engineer with a good knowledge of differential equations can view sensor data and decide if the motor is dominated by Coulomb friction, viscous friction, etc.



- (a) Classify the previous equation by picking the relevant qualifiers from the following list.

Linear	Homogeneous	Constant-coefficient	1st-order	Algebraic
Nonlinear	Inhomogeneous	Variable-coefficient	2nd-order	Differential

- (b) Using *separation of variables and integration*, find the solution for $\omega(t)$ in terms of I , T_f , and ω_0 [the *initial value* of ω , i.e., $\omega(t=0)$].

Result:

$$\omega(t) = \text{[Yellow Box]}$$

- (c) A motor having an inertia (with attached rotor) of $I = 300 \text{ g}\cdot\text{cm}^2$ is spinning with $\omega = 60 \text{ rad/sec}$. The power is then shut off and the motor spins down with the time-history shown in the previous graph. Approximate T_f , the Coulomb friction torque constant.

Result:

$$T_f \approx 0.5 \text{ milliNewtons}\cdot\text{m}$$

1.12 System identification for a 1st-order dynamic system (viscous damping)

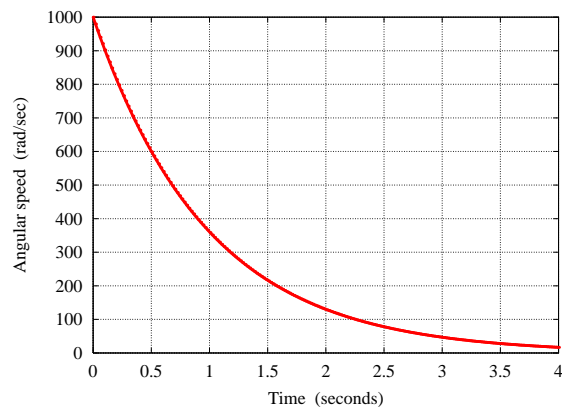
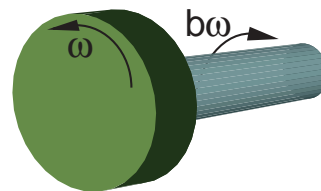
The figure to the right shows a freely-spinning (non-driven) motor whose laboratory data correlates with the mathematical function

$$\omega(t) = \omega_0 e^{-\frac{t}{\tau_c}}$$

where ω_0 is the initial angular spin rate and τ_c is called the *time constant*. This mathematical function suggests a 1st-order ODE of the form

$$I \frac{d\omega}{dt} + b\omega = 0$$

where I is a constant (the moment of inertia of the motor and its rotor and attachments about the motor's axis) and b is the linear viscous damping constant.



- (a) Classify the previous equation by picking the relevant qualifiers from the following list.

Linear	Homogeneous	Constant-coefficient	1st-order	Algebraic
Nonlinear	Inhomogeneous	Variable-coefficient	2nd-order	Differential

- (b) A motor is spinning with $\omega = 1000 \text{ rad/sec}$. The power is shut off and the motor spins down with the time-history shown. Use the laboratory data to approximate the time-constant τ_c .³

Result:

$$\tau_c \approx 0.98 \text{ sec}$$

- (c) A motor having an inertia (with attached rotor) of $I = 300 \text{ g}\cdot\text{cm}^2$ is spinning with $\omega = 1000 \text{ rad/sec}$. The power is then shut off and the motor spins down. By observing the time-history of the angular speed of the motor, an engineer approximates the time-constant to be $\tau_c = 4 \text{ sec}$. Assuming that linear viscous damping is the only relevant factor in the motor's spin-down, determine the linear viscous damping constant b .⁴

Result:

$$b = 75 \frac{\text{g}\cdot\text{cm}^2}{\text{sec}}$$

³The *time constant* τ_c is defined in Section 4.5.

⁴Hint: Use *separation of variables and integration*, to solve for $\omega(t)$ in terms of I , b , and ω_0 [the *initial value* of ω].

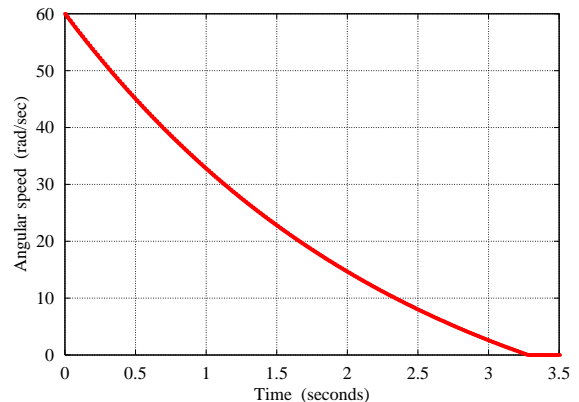
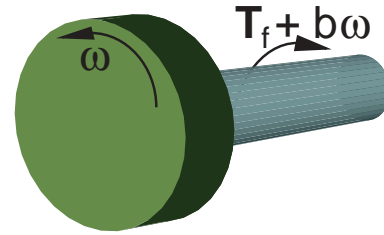
1.13 System identification for a 1st-order dynamic system (Coulomb friction and viscous damping)

The figure to the right shows a freely-spinning (non-driven) motor whose laboratory data suggests that both Coulomb friction and linear viscous damping are acting on the motor. The equation governing ω (the motor's angular speed) is

$$I\dot{\omega} + b\omega = -T_f$$

- I is the moment of inertia of the motor and its attachments about the motor's axis
- b is the linear viscous damping constant
- T_f is the constant torque exerted by Coulomb friction

The indication that both Coulomb friction and viscous damping are present is that at large values of ω , viscous damping dominates and the motor's speed decreases exponentially whereas at small values of ω , Coulomb friction dominates and the motor's speed decreases linearly.



- (a) Classify the previous equation by picking the relevant qualifiers from the following list.

Linear	Homogeneous	Constant-coefficient	1st-order	Algebraic
Nonlinear	Inhomogeneous	Variable-coefficient	2nd-order	Differential

- (b) Use **separation of variables**, show the differential equation can be written as

$$\frac{I d\omega}{b\omega + T_f} = -dt$$

Use **integration** to solve for $\omega(t)$ in terms of I , b , T_f , and ω_0 (the initial value of ω). It may be helpful to make a substitution ($u = \dots$) to do the integral.

Result:

$$\omega(t) = \text{[Yellow Box]}$$

Note: Substituting $T_f=0$ gives $\omega(t) = \omega_0 e^{-\frac{b}{I}t}$ which is useful for Homework 1.12. Doing a Taylor-series expansion about $b=0$ in the solution for $\omega(t)$ gives $\omega(t) = \omega_0 - \frac{T_f}{I}t$ which verifies the solution in Homework 1.11.

- (c) A motor having an inertia (with attached rotor) of $I = 1.0 \text{ kg}\cdot\text{m}^2$ is spinning with an initial angular speed of $\omega_0 = 60 \text{ rad/sec}$. The power is then shut off and the motor spins down with the time-history shown above. Using the laboratory data, form two equations that suffice to approximate b and T_f . Two convenient points that produce accurate results are $\omega(t=0.5) \approx 45$ and $\omega(t=2.5) \approx 8$.⁵

Result:

$$\begin{aligned} \left(60 + \frac{T_f}{b}\right) * e^{-0.5b} - \frac{T_f}{b} - 45 &= 0 \\ \left(60 + \frac{T_f}{b}\right) * e^{-2.5b} - \frac{T_f}{b} - 8 &= 0 \end{aligned}$$

- (d) Classify the previous set of equations by picking the relevant qualifiers from the following list.

⁵To get two equations that accurately approximate b and T_f , two data points are needed. Since the magnitude of the resisting torque scales with $T_f + b\omega$, the first data point is at a small value of t when the motor is spinning quickly (ω is large) and linear viscous damping ($b\omega$) dominates. The second data point is at a larger value of t when the motor is spinning slowly (ω is small) and Coulomb friction (T_f) dominates.

Uncoupled	Linear	Homogeneous	Algebraic
Coupled	Nonlinear	Inhomogeneous	Differential

(e) With a pencil, paper, and a simple calculator, can *you* find numerical values for b and T_f that satisfy both equations? **Yes/No** (pick one).

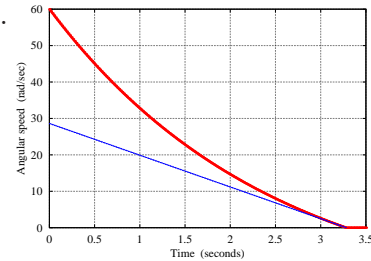
If not, what qualifiers from the previous list are impeding your progress?

(f) Find b and T_f using the technique described in Section 2.12.1. Submit a printed copy of your **MotionGenesis** or Matlab file and use the `Exp()` function for e .

Result:

$$b = 0.41 \frac{\text{kg}\cdot\text{m}^2}{\text{sec}} \qquad T_f = 8.72 \text{ N}\cdot\text{m}$$

Note: A good approximation for T_f can be made using the slope of the tangent line as $\omega \rightarrow 0$.



1.14 Optional**: Determining damping and friction constants from experimental data.

The following ODE governs the angular speed ω of a motor whose motor and attached rotor have a moment of inertia of $I = 1.0 \text{ kg m}^2$. The associated graph shows the time-history of ω for a spin-down test corresponding to an initial value of $\omega(t = 0) = 64 \text{ rad/sec}$.

The following laboratory data is available for $\omega(t)$.

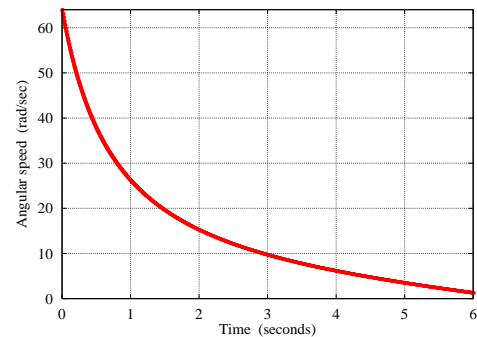
Time (sec)	Angular speed ω (rad/sec)
0.5	37.74139
2.0	15.23263
4.0	6.147114

After using separation of variable and integration to solve the ODE for $\omega(t)$, use the laboratory data to approximate b , c , and T_f .

Result:

$$T_f = 2.0 \text{ N m} \qquad b = 0.04 \text{ N m sec} \qquad c = 0.02 \text{ N m sec}^2$$

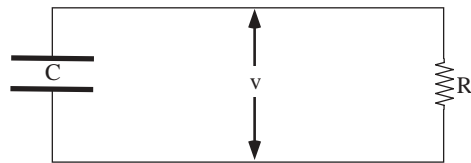
$$I \frac{d\omega}{dt} = -(T_f + b\omega + c\omega^2)$$



1.15 Solving a 1st-order ODE associated with an electrical system

The voltage v in an “R-C” circuit consisting of a linear resistor with resistance r and a linear capacitor with capacitance c is governed by the ODE

$$c\dot{v} + \frac{1}{r}v = 0$$



(a) Classify the previous equation by picking the relevant qualifiers from the following list.

Linear	Homogeneous	Constant-coefficient	1st-order	Algebraic
Nonlinear	Inhomogeneous	Variable-coefficient	2nd-order	Differential

(b) Given an initial voltage $v(t=0) = v_0$, determine $v(t)$.

Result:

$$v(t) = \text{[Yellow Box]}$$

(c) Given that the current i passing through the resistor is $i = \frac{v}{r}$ and $v_0 = 120$ volts, determine values for r and c such that a current of 2.0 milliAmps and 40 volts is produced at $t = 2.0$ seconds.

Result:

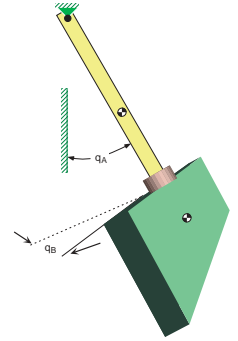
$$r = 20 \text{ kohms (20,000 ohms)} \qquad c = 91 \text{ microFarads (9.1} \times 10^{-5} \text{ Farads)}$$

1.16 Optional**: Numerical solution of ordinary differential equations

The equations of motion for the system described in Section 2.14 are

$$\ddot{q}_A = \frac{2 [508.89 \sin(q_A) - \sin(q_B) \cos(q_B) \dot{q}_A \dot{q}_B]}{-21.556 + \sin(q_B)^2}$$

$$\ddot{q}_B = -\sin(q_B) \cos(q_B) \dot{q}_A^2$$



- (a) Classify the previous set of equations by picking the relevant qualifiers from the list below.

Uncoupled	Linear	Homogeneous	Constant-coefficient	1st-order	Algebraic
Coupled	Nonlinear	Inhomogeneous	Variable-coefficient	2nd-order	Differential

- (b) Using **MotionGenesis** and/or Matlab (see Section 2.14), find the maximum absolute value of $q_B(t)$ for $0 \leq t \leq 10$. Do this for $q_B(0)=1.0^\circ$ and for two different starting values of q_A , namely $q_A(0)=45^\circ$ and $q_A(0)=90^\circ$. Submit a printed copy of your **MotionGenesis** or Matlab file.

Result:

- Maximum absolute value of $q_B(t)$ when $q_A(0)=45^\circ$: $q_B(t=1.66) = \text{[yellow box]}^\circ$
- Maximum absolute value of $q_B(t)$ when $q_A(0)=90^\circ$: $q_B(t=8.18) = \text{[yellow box]}^\circ$

Homework 2. Chapters 5, 7.
Uncoupled, linear, constant-coefficient, homogeneous, 2nd-order, ODEs

2.1 Euler's formula and trigonometry functions

Show every step to express the right-hand side of the following expressions in terms of trigonometric functions of the real scalar variable θ (without the imaginary number i).

$$\frac{e^{i\theta} + e^{-i\theta}}{2} = \text{[Yellow Box]} \qquad \frac{e^{i\theta} - e^{-i\theta}}{2i} = \text{[Yellow Box]}$$

2.2 Undamped free vibrations

A system whose response is governed by the linear, second-order, ODE¹

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2y = f(t)$$

is said to be performing **undamped free vibrations** if $\zeta=0$ (**undamped**) and $f(t)=0$ (**free**). Section 7.5 shows $y(t)$ can be expressed in terms of the yet-to-be-determined constants A and B as

$$y(t) = A \sin(\omega_n t) + B \cos(\omega_n t)$$

- (a) Suppose the initial values of $y(t)$ and $\dot{y}(t)$ are known. Determine A and B and rewrite the previous equation in terms of $y(0)$ and $\dot{y}(0)$.

Result:

$$y(t) = \left[\frac{\dot{y}(0)}{\omega_n} \right] \sin(\omega_n t) + [y(0)] \cos(\omega_n t)$$

- (b) Express the amplitude C and the phase angle ϕ in terms of $y(0)$ and $\dot{y}(0)$ when $y(t)$ is expressed as $y(t) = C \cos(\omega_n t + \phi)$.

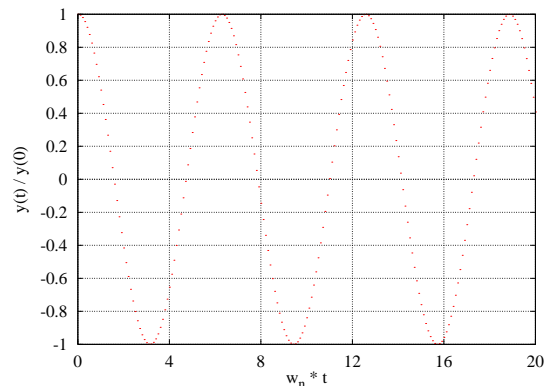
Result:

$$C = \sqrt{\left[\frac{\dot{y}(0)}{\omega_n} \right]^2 + [y(0)]^2} \qquad \phi = \text{atan2} \left(\frac{-\dot{y}(0)}{\omega_n}, y(0) \right)$$

Assuming $\dot{y}(0)=0$, sketch $\frac{y(t)}{y(0)}$ versus $\omega_n t$ for $0 \leq \omega_n t \leq 20$.^a

- (c) The **period of a function** is denoted τ_{period} and is defined in Section 8.1. Mark τ_{period} on your sketch.

^aYou may plot with Alplot, Matlab, or by hand.



- (d) Determine the period of the function $y(t) = C \cos(\omega_n t + \phi)$.

Result:

$$\tau_{period} = \text{[Yellow Box]}$$

¹ ω_n and ζ are constants called the **natural frequency** and **damping ratio**, respectively.

2.3 Underdamped free vibrations

A system whose response is governed by the linear, second-order, ODE

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2y = f(t)$$

is said to be performing **underdamped free vibrations** if $0 < \zeta < 1$ (**underdamped**) and $f(t)=0$ (**free**). One quantity that is helpful for answering the following questions is the **damped natural frequency**, $\omega_d \triangleq \omega_n \sqrt{1 - \zeta^2}$.

- (a) After assuming a solution $y(t) = Ce^{pt}$ where C and p are constants, **show every step** that proves $y(t)$ can be expressed in terms of the yet-to-be-determined constants A and B as

$$y(t) = [A \sin(\omega_d t) + B \cos(\omega_d t)] e^{-\zeta\omega_n t}$$

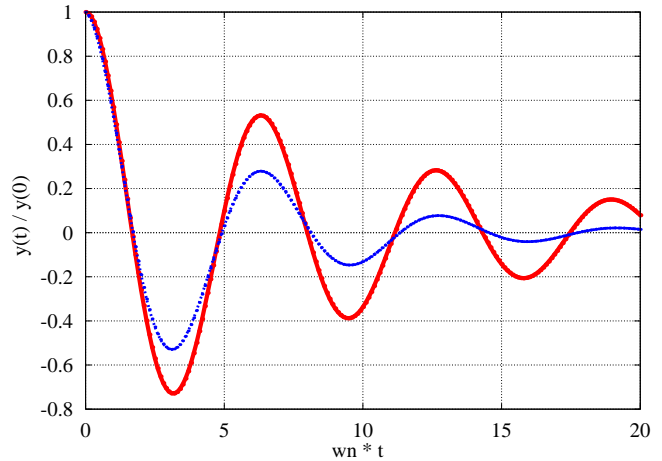
- (b) Suppose the initial value of $y(t)$ and $\dot{y}(t)$ are known. Determine A and B and rewrite the previous equation in terms of $y(0)$ and $\dot{y}(0)$.

Result:

$$y(t) = \left\{ \left[\frac{\dot{y}(0)}{\omega_d} + \frac{\zeta y(0)}{\sqrt{1-\zeta^2}} \right] \sin(\omega_d t) + y(0) \cos(\omega_d t) \right\} e^{-\zeta\omega_n t}$$

- (c) Assuming $\dot{y}(0) = 0$, show $\frac{y(t)}{y(0)} = \left\{ \frac{\zeta}{\sqrt{1-\zeta^2}} \sin(\omega_d t) + \cos(\omega_d t) \right\} e^{-\zeta\omega_n t}$

The plot to the right shows $\frac{y(t)}{y(0)}$ versus $\omega_n t$ for $0 \leq \omega_n t \leq 20$. **Label** the curve corresponding to $\zeta=0.1$ and $\zeta=0.2$.



The **period of a vibration** is denoted τ_{period} and is defined in Section 8.1. Mark τ_{period} on your sketch by first marking the local maxima (or minima). In other words, mark the points on the curve where $\dot{y}(t)=0$ and $\ddot{y}(t) < 0$ and label the time interval between two successive local maxima (or minima) as τ_{period} .

- (d) In view of your sketch, complete the following statement:
Increasing the damping ratio from $\zeta=0.1$ to $\zeta=0.2$ causes $y(t) \rightarrow 0$ **slower/faster**.
- (e) Express C and ϕ in terms of $y(0)$ and $\dot{y}(0)$ when $y(t)$ is expressed $y(t) = C \cos(\omega_d t + \phi) e^{-\zeta\omega_n t}$.

Result:

$$C = \sqrt{\left[\frac{\dot{y}(0) + \zeta\omega_n y(0)}{\omega_d} \right]^2 + [y(0)]^2} \quad \phi = \text{atan2} \left(- \left[\frac{\dot{y}(0) + \zeta\omega_n y(0)}{\omega_d} \right], y(0) \right)$$

- (f) Assuming $\omega_n = 1 \frac{\text{rad}}{\text{sec}}$, find ω_d and τ_{period} for the values of ζ in the following table.

	$\zeta=0$	$\zeta=0.1$	$\zeta=0.3$	$\zeta=0.5$	$\zeta=1.0$	$\zeta=2.0$
ω_d (rad/sec)	1.0				0.0	imaginary
τ_{period} (sec)	6.28				∞	imaginary

- (g) In view of your table, complete the following statements:
Increasing the damping ratio from $\zeta=0$ to $\zeta=0.5$ corresponds to a **smaller/slightly smaller/no change/slightly larger/larger** damped natural frequency and a **smaller/slightly smaller/no change/slightly larger/larger** period of vibration.

2.4 Critically damped free vibrations

A system whose response is governed by the linear, second-order, ODE

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2y = f(t)$$

is performing **critically damped free vibrations** if $\zeta=1$ (**critically damped**) and $f(t)=0$ (**free**).

- (a) Verify that the solution

$$y(t) = (A + Bt) e^{-\omega_n t}$$

satisfies the differential equation. Note: Do not derive this solution; just ensure that substituting the solution into the differential equation produces a sensible result. It is worth mentioning that the definition of **period of a function** τ_{period} in Section 8.1. is meaningless for critically damped and overdamped vibrations.

- (b) Suppose the initial values of $y(t)$ and $\dot{y}(t)$ are known. Determine A and B and rewrite the previous equation in terms of $y(0)$ and $\dot{y}(0)$.

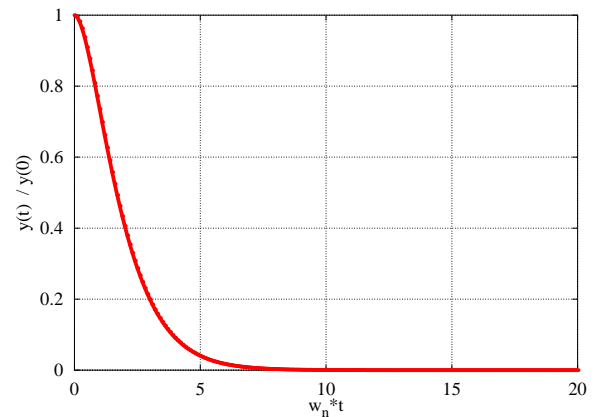
Result:

$$y(t) = \{ y(0) + [\dot{y}(0) + \omega_n y(0)] t \} e^{-\omega_n t}$$

The plot to the right shows $\frac{y(t)}{y(0)}$ versus $\omega_n t$ for $0 \leq \omega_n t \leq 20$ when $\dot{y}(0) = 0$.

- (c) In view of this plot and the plot in Homework 2.3, complete the following statement:

Increasing the damping ratio from $\zeta=0.2$ to $\zeta=1.0$ causes $y(t) \rightarrow 0$ **slower/faster**.



2.5 Overdamped free vibrations

A system whose response is governed by the linear, second-order, ODE

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2y = f(t)$$

is performing **overdamped free vibrations** if $\zeta > 1$ (**overdamped**) and $f(t)=0$ (**free**).

Two quantities which are helpful in the questions that follow are p_1 and p_2 , defined as

$$\begin{aligned} p_1 &\triangleq -\omega_n(\zeta - \sqrt{\zeta^2 - 1}) \\ p_2 &\triangleq -\omega_n(\zeta + \sqrt{\zeta^2 - 1}) \end{aligned}$$

- (a) After assuming a solution of the form $y(t) = Ce^{pt}$ where C and p are constants, show **every step** that proves $y(t)$ can be expressed in terms of the yet-to-be-determined constants A and B as

$$y(t) = A e^{p_1 t} + B e^{p_2 t}$$

It is worth noting that there exists at most one value of t such that $\dot{y}(t)=0$. Therefore, the definition of **period of a function** τ_{period} in Section 8.1. is meaningless for overdamped vibrations.

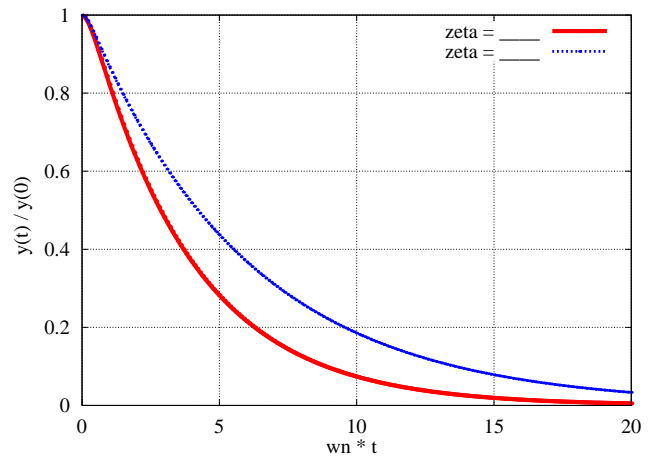
- (b) If the initial value of $y(t)$ and $\dot{y}(t)$ are known, the previous equation can be written in terms of $y(0)$ and $\dot{y}(0)$ as (you do not need to show this)

$$\begin{aligned} y(t) &= \left[\frac{\dot{y}(0) - p_2 y(0)}{p_1 - p_2} \right] e^{p_1 t} - \left[\frac{\dot{y}(0) - p_1 y(0)}{p_1 - p_2} \right] e^{p_2 t} \\ &= \left[\frac{\dot{y}(0) - p_2 y(0)}{2\omega_n \sqrt{\zeta^2 - 1}} \right] e^{p_1 t} - \left[\frac{\dot{y}(0) - p_1 y(0)}{2\omega_n \sqrt{\zeta^2 - 1}} \right] e^{p_2 t} \end{aligned}$$

The plot to the right shows $\frac{y(t)}{y(0)}$ versus $\omega_n t$ for $0 \leq \omega_n t \leq 20$ when $\dot{y}(0)=0$.

Label the curve corresponding to $\zeta=2.0$ and $\zeta=3.0$.

In view of this plot and the plot in Homework 2.4, complete the following statement:
Increasing the damping ratio from $\zeta=1$ to $\zeta=2$ or $\zeta=3$ causes $y(t) \rightarrow 0$ **slower/faster**.



2.6 Unstable PD feedback control of a mass-spring-damper system

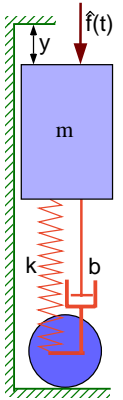
The ODE governing a “mass-spring-damper” system such as the one to the right can be written as the ODE shown below-left (you do not have to show this).^a

The point of this problem is to investigate this system’s response for “**proportional-derivative (PD) feedback-control**” of the form^b $f(y, \dot{y}) = 4\dot{y} + 10y$.

$$\ddot{y} + 2\dot{y} + 2y = f(y, \dot{y}) \quad \Rightarrow \quad \ddot{y} + -2\dot{y} + -8y = 0$$

^aThe downward displacement shown in the figure corresponds to a **positive** value for y .

^bOne does **not** usually choose a controller that de-stabilizes a system.



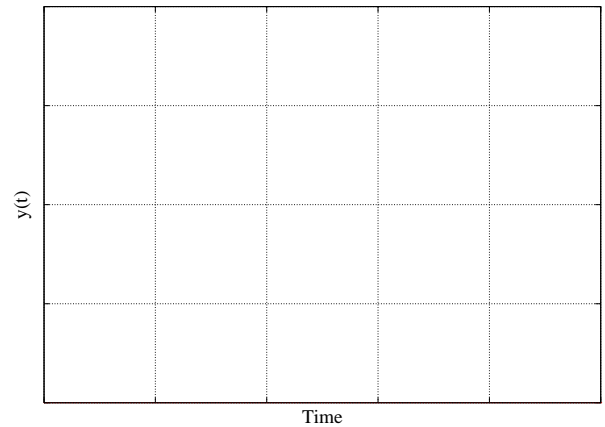
Given the initial values $y(0) = 0.1$ and $\dot{y}(0) = 0$, **show every step** that solves the ODE for $y(t)$ (i.e., find a function $y(t)$ in terms of numbers and time).

Next, plot $y(t)$ showing the initial values of y and \dot{y} and the general shape of the curve.

List physical limits of this control system.

- **Position:** The suspension system quickly hits the **ground/sky** (y grows exponentially).
- **Velocity:** Exceeds the **speed of light** ($c \triangleq 299,792,458 \frac{\text{m}}{\text{sec}}$) within \approx sec.
- **Acceleration:** Exceeds \approx g's within 0.71 sec.
- **Force:** The control force **decays/grows** exponentially (unrealistic).

Result: $y(t) =$



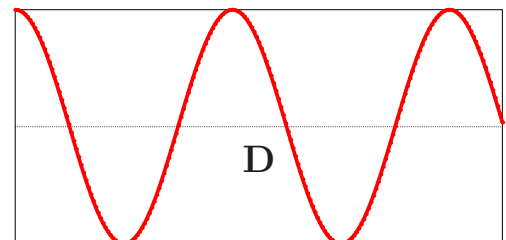
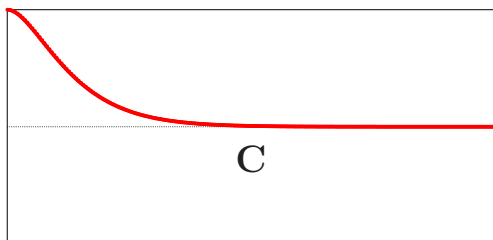
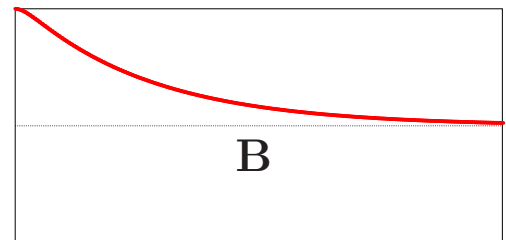
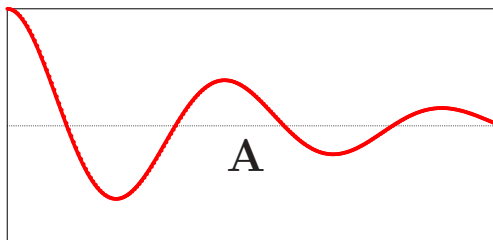
2.7 System response and damping

A dynamic system is governed by the second-order, linear, ODE

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2y = 0$$

Referring to the following graphs, complete the table to the right with **A, B, C, or D**. Note: $\omega_n = 1$ was used for all graphs.

Undamped	<input type="checkbox"/>
Underdamped	<input type="checkbox"/>
Critically damped	<input type="checkbox"/>
Overdamped	<input type="checkbox"/>
Shortest settling time	<input type="checkbox"/>



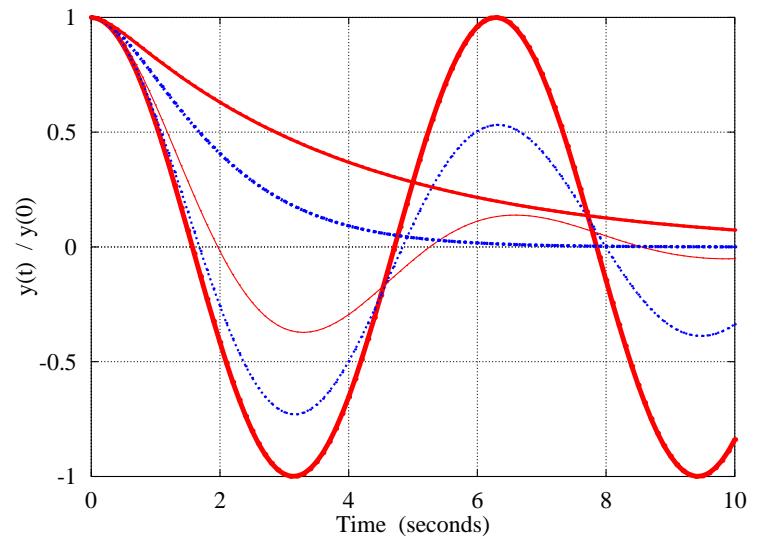
2.8 System response for various values of ζ

A system is governed by the 2nd-order, linear, ODE

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = 0$$

- (a) With $\dot{y}(0)=0$ and $\omega_n = 1 \frac{\text{rad}}{\text{sec}}$, $\frac{y(t)}{y(0)}$ was plotted for 10 seconds with various values of ζ as shown.

Label each curve on the plot with $\zeta=0$, $\zeta=0.1$, $\zeta=0.3$, $\zeta=1.0$, or $\zeta=2.0$.



- (b) Associate each of the following values ζ with **undamped**, **underdamped**, **critically damped**, or **overdamped** and circle the value of ζ that corresponds to $y(t) \rightarrow 0$ as fast as possible.

$\zeta=0$	$\zeta=0.2$	$\zeta=0.7$	$\zeta=1.0$	$\zeta=4.0$