

Chapter 1

What is a dynamic system?

A *dynamic system* is a system whose behavior is time-dependent.¹ For example, the motion of an aircraft, automobile, spacecraft, animal, or electron is a dynamic system. Dynamic systems cover a broad range of subjects including: vibrations, dynamics, control-system design, fluids, heat-transfer, circuits, sensors, mechatronics, pneumatics, biology, chemistry, economics, and transportation.

The purpose of this book is to provide mechanical, aerospace, and biomechanical engineers with useful tools for predicting, controlling, and designing dynamic systems governed by ordinary differential equations.² In addition, this book is fundamental for subsequent courses in

- Control systems
- Finite element analysis (FEA)
- Advanced vibrations, modal analysis, and input shaping
- Biomechanics, multibody dynamics, robotics, and mechatronics

1.1 MIPS I

One procedure for studying dynamic systems is summarized with the acronym MIPS I.

- **M**odel: Capture the essential components of the physical system being analyzed and draw a simple sketch of the model.
- **I**dentifiers: Name and label relevant parts, e.g., bodies, lengths, and angles, by introducing symbols such as m , g , L , θ , etc. Analytically or empirically determine values for physical constants.
- **P**hysics: Using physical principles, e.g., Newton's laws or the Kirchoff Current Law, formulate equations which relate the identifiers and govern the behavior of the system.
- **Simplify and solve.** When useful, simplify the mathematical analysis, e.g., with small angle or linear approximations. Produce numerical or analytical (closed-form) solutions for the unknown identifiers.
 - Linearization of nonlinear differential equations
 - Separation of variables, assumed solutions, numerical solutions of ordinary differential equations
 - Solution techniques for linear and nonlinear algebraic equations
 - Eigenanalysis of coupled sets of differential equations
- **Interpret, design, and control:** Using numbers, plots, animation, force-feedback, etc., generate results that can be easily interpreted, preferably by a non-technical person. Then, use the results to improve the system's design or behavior, e.g., with a PID control system.

¹A system whose behavior does *not* depend on time is called a *static system*.

²Discrete-time systems and systems governed by partial differential equations are not discussed in this book.

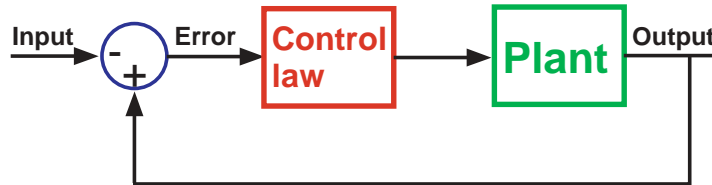
1.2 Important concepts in dynamic systems

The building blocks of dynamic systems include differential equations, complex numbers, Laplace transforms, eigen-analysis, Newton's laws, circuit analysis, and block diagrams. Examples of these building blocks are summarized below:

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = f(t)$$
$$e^{i\theta} = \cos(\theta) + i \sin(\theta)$$

$$G(s) = \frac{1}{m s^2 + b s + k}$$
$$A v = \lambda v$$

$$\mathbf{F} = m \mathbf{a}$$
$$v = i R$$



1.3 Objectives of this textbook

The high-level objectives of this textbook are to:

- Model, introduce mathematical identifiers, analyze, and interpret dynamic systems
- Understand the roles of analytical methods and computational tools, including designing feedback control systems with professional software including Matlab, **M**otion**G**enesis, and Working Model
- Develop intuition about the time-dependent nature of dynamic systems
- Gain physical insights into basic principles with experiments
- Develop a hands-on, minds-on, can-do attitude

The specific skills that the reader will acquire through reading the text, following the examples, doing the homework, and trying the demonstrations and labs are to:

- Form, classify, and solve differential equations for mechanical and electromechanical systems
- Determine stability of a system (uncontrolled and controlled)
- Size springs, dampers, motors, resistors, capacitors, and inductors to change a system's behavior, e.g., to modify natural frequency or damping
- Understand the effect of forcing frequency on a system's behavior, e.g., the effect of earthquakes on buildings and the effect of music signals on stereo speakers
- Regulate mechanical and electromechanical systems with relatively simple controllers
- Design a PID controller to meet design specifications, e.g., maximum overshoot, settling time, rise time, etc.
- Predict the behavior of electromechanical systems using Newtonian mechanics and circuit analysis
- Use simple s -domain tools to design low-pass and high-pass filters for sensors
- Find a nominal (desired) solution
- Linearize nonlinear differential equations about a nominal solution
- Perform vibration analysis
- Develop a clear physical and mathematical understanding of eigenvalues and eigenvectors
- Control multiple degree-of-freedom systems with state-space and pole-placement


Chapter 2

Math review

2.1 Why math is important

Math is a foundation for science, medicine, engineering, construction, and business. Math provides **concepts** (pictures, words, and ideas), **calculations** (mathematical operations, symbols, equations, and definitions), and **context** (situations in which the concepts and calculations are relevant and useful). More generally, math is a language and set of rules and conventions that allow humans to count, quantify, calculate, manipulate, relate, define, extrapolate, and abstract “stuff”.¹

Math is critical for precise understanding of many concepts.² Advances in math depend on pictures³ words, symbols, equations, and definitions. For example, consider the following **definition** of π .

Object	Example	Approximate age of human comprehension
Picture		Toddlers
Spoken word	“circle”	Pre-school
Written word	“circle”, “diameter”, “circumference”	Elementary school
Symbol	d for diameter, c for circumference	Middle school
Equation	$c = \pi d$	Middle school
Definition	$\pi \triangleq \frac{c}{d}$	College / graduate school

2.2 A brief history of mathematics

Advances in mathematics and technology are usually made at a painstakingly slow pace, with small sparks of individual brilliance that are accompanied by good luck or divine inspiration. Over the past millennia, various cultures have produced groups of gifted individuals that have had a significant impact on modern mathematics, engineering, and technology, including: Egypt (3000 BC), Babylon (2000 BC), Greece and China (500 BC), India (500 AD), Africa (800 AD), Europe and Asia (1500 AD), United States and Soviet Union (1900 AD), and Worldwide Web (2000⁺ AD).

2.2.1 Real number systems

- **Egyptians (3000 BC):** Egyptians used hieroglyphics (picture writing) for numerals. The system was based on 10, but did not include a zero or the principle of place value. The (approximate) hieroglyphic symbols shown below combine to depict the number 1,326 as | @@@ ^^ //////.

¹For example, the “idea” of **value** (answering “**how much something is worth**”) is quantified through money.

²Certain concepts are difficult to precisely define. For example, physicists call mass “**fundamental**” because it currently eludes mathematical definition and because humans have an inherent sense of mass.

³**Art** is *not* reserved for the sophisticated and highly educated who have the knowledge and historical context to understand and appreciate it. Appreciation for shapes, colors, and emotional expression in art is available to humans on a basic (almost primitive and subconscious) level.

1	/	Stroke
10	ˆ	Arch
100	@	Coiled Rope
1,000		Lotus Flower
10,000	>	Finger
100,000	~	Tadpole

- **Babylonians (2000 BC):** Developed a number system based on 60. Their number system was more consistent and structured than the Egyptian system and was simpler for mathematical calculations. The number of seconds in a minute (60) and number of minutes in an hour (60) are a consequence of the Babylonian number system.
- **Hittites in Mesopotamia (2000-700 BC):** Developed the precursor to the Roman numeral system e.g., their first four numbers were I, II, III, and IIII (note how I looks like a finger or a stick).
- **Romans (500 BC):** The early Roman system was based on the Hittite system, e.g., its first four numbers were I, II, III, and IIII. The Roman system had special symbols for 5 (V), 10 (X), 50 (L), 100 (C), 500 (D), and 1000 (M). The early Roman system only used **addition** (not subtraction) thus 4 was IIII (not IV), 6 was VI, 9 was VIIII (not IX), and 11 was XI. The late Roman numeral system was invented in France in 1500 AD by clock and watch makers to save space on clocks. The Roman system was used throughout the Middle Ages and by law was the only acceptable system in Europe until 1300 AD.
- **Greeks (500 BC):** Developed a system based on 10. The first nine letters of their alphabet represented the numbers 1-9. The next nine letters stood for tens, from 10 through 90. The last nine letters were for hundreds, 100 through 900. The Greeks combined their symbols like the Egyptians.
- **Hindus (200 BC - 700 AD):** Around 200-300 BC, the Hindus in India used a system based on 10. They had symbols for each number from 1 to 9 and a symbol for each power of 10. Thus a Hindu wrote “1 sata, 3 dasan, 5” to write the number 135. Around 600 AD the Hindus found a way to eliminate place names. They invented the symbol “sunya” (meaning empty), which we call zero. With this symbol they could write “105” instead of “1 sata, 5”.

The use of *negative numbers* in solving problems can be traced as far back as the Indian Brahmagupta (7th century AD) who used zero and negative numbers in his algebraic work. He even gave the rules for arithmetic, e.g., “a negative number divided by a negative number is a positive number.” This may be the earliest [known] systemization of negative numbers as entities in themselves.

Seventeenth century Europeans were the first to broadly accept **negative** numbers. Before 1700 AD, negative numbers were treated with great suspicion. The ancient Chinese calculated with **red** (positive) and **black** (negative) rods (opposite to today’s accounting practices), but, like other cultures, they did not accept a negative number as a solution to a problem. Instead, the problem was stated so its result was a positive quantity.

Note: A 4-year old child’s answer to the problem of 3 apples - 5 apples was a profoundly definite “*you can not do that*”. Humans use negative numbers to convey **opposites**, e.g., negative/positive temperatures convey cold and hot, debt is negative whereas assets are positive, electrons and protons have positive and negative charge, etc.

- **Mayans (700-800 AD):** The Mayan civilization invented a highly sophisticated vertical number system that used positional base notation as well as zero. Unfortunately, their knowledge and civilization did not survive the centuries and Mayan mathematics has had little impact on modern mathematics or technology.
- **Arabs (700-800 AD):** Muhammad ibn Musa Al-Khwarizmi (780-850 AD) learned the Hindu number system and extended it by using **zero** as a place holder in positional base notation. In 800 AD, he wrote a book that was later translated into Latin around 1100 AD. Many mathematicians consider the Hindu-Arabic number system the world’s greatest mathematical invention because it introduced the idea of place value and zero.

- **Europeans (1000-1200 AD):** The Latin translation of the Hindu-*Arabic number system*, the encouragement of the astute mathematician Gerbert (who later was made Pope Sylvester II) in 980 AD, and the notational work of Leonardo of Pisa (Fibonacci) in 1202 AD, resulted in the number system being used worldwide today. Because of political and cultural conflicts, widespread use of the Arabic number system was delayed until 1500 AD.⁴
- **Europeans (1700 AD):** Gottfried Wilhelm Leibniz (1646-1716) developed the *binary number system* (now used by most computers) in which he interpreted 1 for God and 0 for “the void”.
- **Fractions** were used by the Egyptians (3000 BC), Babylonians (2000 BC), and Greeks (500 BC). Representing fractions with two stacked integers was essentially due to the Hindus (628 AD). The horizontal fraction bar (called the *vinculum*) was introduced by the Arabs around 1200 AD. Soon thereafter, the diagonal fraction bar (called a *solidus* or *virgule*) was used for print because of typographical difficulties with the horizontal fraction bar.
- **Percent symbols** (denoting $\frac{1}{100}$) were introduced in an anonymous Italian manuscript in 1425 AD.
- **Decimal points** were first written as a blank space by the Persian astronomer Al-Kashi (1426). In 1530, Christoff Rudolff used a vertical bar exactly as we use a decimal point today. Before 1617 when Napier used both a comma and period to separate units and tenths, several other notations were in use, e.g., Simon (1585), Viete (1600), and Kepler (1616). Modern monetary and measurement systems (e.g, the SI system) take advantage of the conveniences afforded by the base-10 number system and decimal point [denoted with either a period (.) or comma (,)].

2.2.2 Imaginary numbers, complex numbers, and quaternions

- Stereometria of Heron of Alexandria (circa 50 AD) first noticed imaginary numbers when he saw that $\sqrt{81-144}$ could not be computed, so he switched it to $\sqrt{144-81}$.
- Arithmetica of Diophantus (c. 275 AD) noticed imaginary numbers when he attempted to compute the sides of a right triangle with a perimeter of 12 and an area of 7. He found it necessary to solve the equation $24x^2 - 172x + 336 = 0$. He did not understand that the equation had complex roots (there are no real triangles with a perimeter of 12 and an area of 7).
- Mahavira (c. 850 AD) stated that a negative number is not a square and does not have a square root.
- Bhaskara (c. 1150 AD) described imaginary numbers using language similar to Mahavira.
- The Arabs and Persians paid no special attention to the subject.
- Pacioli (1494 AD) stated in his Summa that the quadratic equation $x^2 = c = bx$ cannot be solved unless $b^2/4$ is greater than or equal to c . He recognized the impossibility of finding $\sqrt{-1}$.
- Cardan (1545 AD) was the first to use the square root of a negative number in a computation. The problem was to divide 10 into two parts whose product was 40. He determined the number to be $5 + \sqrt{-15}$ and $5 - \sqrt{-15}$ and proved by multiplication that his results were correct.
- Stevin (1585 AD) spoke of the difficulty of working with imaginary numbers but only remarked that that the subject was not mastered.
- Wallis (1673 AD) was the first to give a graphical representation of *imaginary numbers* in a plane (similar to positive and negative numbers on a line). He stated that the square root of a negative number was thought to imply the impossible, but the same might be said of a negative number.
- Newton’s work (1685 AD) with imaginary numbers was confined to the number of roots of an equation.
- Jean Bernoulli (1702 AD) related the atan function and the logarithm of an imaginary number.

⁴Until *Pope Sylvester’s* Latin translation of the Arabic number system, the Europeans resisted the number 0 and using this great “unholy infidel” number system. The Europeans raised philosophical and religious objections to 0 (e.g., 0 cannot exist because there is always God). Adopting the Arabic number system was complicated because of deep animosity between Muslim Arabs and Christian Europeans fueled by the holy wars.

- Cotes (1710 AD) stated that $\log[\cos(x) + i \sin(x)] = ix$.
- Euler (1727 AD) invented the symbol e for the base of natural logs and the corollary of Cotes' formula $e^{ix} = \cos(x) + i \sin(x)$. Euler also invented the notation $i = \sqrt{-1}$.
- Casper Wessil (**1797 AD**), a Norwegian surveyor, presented the modern geometric theory of **complex numbers** and the **complex plane** before the Royal Academy of Denmark.
- Sir William Hamilton (1830 AD) developed a system of hypercomplex numbers called **quaternions** which have four elements $(1, i, j, k)$. Quaternions proved adaptable for operations in three dimensional space but have mostly been superseded by vectors. A quaternion is usually written in the form $\mathbf{q} = a_0 + a_1*i + a_2*j + a_3*k$. The coefficients a_0, a_1, a_2, a_3 are called **Euler parameters** and are still used for 3D descriptions of orientation of rigid bodies. Note: A quaternion is similar to a complex number in that a complex number may be written as $a_0 + a_1*i$ where a_0 and a_1 are real numbers.

2.2.3 Vectors and dyadics

J. Williard Gibbs (1900 AD) developed **vectors** (quantities having magnitude and direction) and **dyads** (quantities with magnitude and two directions). Vectors and dyadics are extensively used in geometry, statics, dynamics, and motion design and analysis.

2.2.4 Matrices and determinants

Matrices are useful for organizing sets of linear equations. The Babylonians (300 BC) studied problems that lead to simultaneous linear equations and some of these are preserved in clay tablets. The practical theories of **matrices** and **determinants** did not develop until nearly **1800 AD**. The idea of a determinant first appeared in Japan (Seki 1683) and Europe (Leibniz 1683) within a few months of each other. In 1750, Cramer gave a general rule for finding a determinant of a $n \times n$ system of equations. In 1772, Laplace discovered a new formula for the expansion of a determinant. In 1773, Lagrange discovered that a 3×3 determinant could be interpreted as the volume of a tetrahedron. The great mathematician Gauss (1801) first used the term “determinant” calculated inverses, and did “Gaussian elimination”. In 1812-1826, Cauchy did a detailed study of determinants, minors, eigenvalues, and diagonalization of a matrix. D’Alembert, Jacques Sturm, Jacobi, and Eisenstein made contributions to determinants, and Cayley (1841) introduced the notation of two vertical lines that denotes determinants in modern work. In 1850, Sylvester termed the word “matrix” and shared his work with Cayley who later (1858) defined the word matrix more precisely. The Jordan canonical form appeared in 1870. Frobenius defined the rank of a matrix and orthogonal matrices in 1878.

2.3 Mathematical operations

1. **Addition:** One way to conceptualize, visualize, and understand adding real numbers is to use a real number line, e.g., a football field. For example, to visualize $3 + 5 = 8$, stand on the 3 yard line with larger numbers to the right and smaller numbers to the left. Then **move to the right** 5 yards to the answer (the 8 yard line). To visualize $8 + -2$, stand on the 8 yard line and **move to the left** 2 yards (adding a negative number).

The $+$ symbol has meaning in a variety of other contexts and its operational significance changes with each context. For example, the addition of two vectors \mathbf{a} and \mathbf{b} is denoted $\mathbf{a} + \mathbf{b}$. The addition of two appropriately sized matrices A and B is denoted $A + B$. The representation of a complex number whose real part is a and imaginary part is $b*i$ is frequently⁵ denoted $a + b*i$. It is worth noting that using the $+$ symbol to represent a complex number is called a “formal sum” because the quantity to the left of the $+$ sign is not of the same type as the quantity to the right of the $+$ sign. In other words, $3 \text{ apples} + 5 \text{ oranges}$ is a formal sum. One way to understand the addition of a real

⁵Complex numbers are sometimes represented as (a, b) . The imaginary number $\sqrt{-1}$ is usually represented by i or j .

number and an imaginary number is to visualize a large checker board with two perpendicular axes, a real axis and imaginary axis. The real axis has larger real numbers to the right and smaller real numbers to the left whereas the imaginary axis has larger imaginary numbers in front (forward) and smaller real numbers behind. To find $3 + 5*i$, start at the intersection of the real and imaginary axis (0,0) and **move to the right** 3 steps and then **move forward** 5 steps.

2. **Subtraction:** Like addition, subtraction can be visualized using a number line. For example, to visualize $8 - 5 = 3$, stand on the 8 yard line with larger numbers to the right and smaller numbers to the left. Then **move to the left** 5 yards to the answer (the 3 yard line). It is the author's opinion that denoting subtraction with a single dash e.g., $8 - 5$, is a convenience that causes significant confusion that can be avoided by denoting $8 - 5$ as $8 + -5$. In other words, one can regard addition as making a number more positive (**moving right** on the number line), whereas subtraction can be regarded as making a number more negative (**moving left** on the number line). It is easier to teach and learn $3 - 5 = -2$ by instead writing it as $3 + -5 = -2$. To visualize $3 + -5 = -2$, stand on the 3 yard line, **move to the left** 5 yards to the answer (the -2 yard line).

Regardless of the notation used for subtraction, one must understand that the answer -2 means "negative 2" and not "minus 2" or "subtract 2". The dash that is typically used to denote subtraction (the *binary minus* - between two symbols) in $8 - 5$ does not have the same meaning as the dash used to denote negation (*unary minus*) in -2. Overloading the dash symbol - to have different meanings in various mathematical contexts is confusing.

Subtraction is negation and addition.

To reduce confusion and substantially speed the learning process, subtraction should be regarded as negation and addition. For example, many digital computers implement subtraction by negation (changing a single bit) and subsequent addition. Symbolic manipulators implement subtraction as negation and addition because it substantially assists in automatic simplification of symbolic expressions. Lastly, there is **no need to memorize formulas involving subtraction**, e.g., formulas for $\sin(a-b)$, $\cos(a-b)$, and $\frac{d(u-v)}{dt}$.

The efficient method for teaching subtraction, called the *addition method* or *Austrian method*:

- Eliminates the unwieldy (and expensive) process of "*borrowing*"
 - Eliminates the need to memorize subtraction tables
 - Relies on a student's ability to add
 - Mimics the teaching of division (which relies on a student's ability to multiply)
3. **Negation:** To add to the confusion of the dash (-) representing both subtraction and negation, there are unresolved order-of-operations conventions involving negation. Different conventions are used with various textbooks, compilers, and mathematical/spreadsheet programs. For example, is

$$-3^2 = (-3)^2 = +9 \qquad \text{or} \qquad -3^2 = (-1) * 3^2 = -9$$

The answer depends on the precedence of unary minus and whether one treats the - as an inherent part of 3, e.g., $(-3)^2 = 9$ or one treats the - as representing multiplication by -1, e.g., $(-1) * 3^2 = -9$. It is the author's opinion that the unary minus represents multiplication by -1 as this parallels other mathematical conventions. For example, the positive number line is extended upwards into the complex plane by multiplication of each positive number by $\sqrt{-1}$. The analogy for negation is that the positive number line can be extended to the left by multiplication of each positive number by -1. Hence, negation is a *special case of multiplication* $(-1)*$.

4. **Multiplication:** The symbols $*$, \cdot , and \times denote multiplication. Multiplication of two integers is a convenient way to represent "multiple additions". For example, $3*5$ represents adding 3 five times, in other words

$$3 * 5 = 3 + 3 + 3 + 3 + 3$$

5. **Division:** Both the horizontal fraction bar (e.g., in $\frac{3}{5}$) and the diagonal fraction bar (e.g., in $3/5$) are used to denote division. The symbols and ideas in division are heavily rooted in **fractions** and the ratio of two **integers**, dating back to the Egyptians (3000 BC), Babylonians (2000 BC), and Greeks (500 BC). Like subtraction, division is extraneous as **division is multiplication and exponentiation with -1**, e.g.,

$$\frac{y}{x} = y * x^{-1}$$

Concomitantly, there is **no need to memorize formulas involving division**, e.g., the formula for $\frac{d}{dt}(\frac{u}{v})$ is easily calculated using product rules and exponent rules for differentiation.

6. **Exponents:** An exponent can be a convenient way to represent a series of multiplications. For example, 3^5 represents multiplying 3 five times, in other words

$$3^5 = 3 * 3 * 3 * 3 * 3$$

This interpretation works well for integer exponents. Some exponents have special names, e.g., the exponents 2 (**squared**), $\frac{1}{2}$ (**square root**), 3 (**cubed**), and $\frac{1}{3}$ (**cube root**).

One dilemma that took centuries to resolve starts as follows:

$$5^2 = 25 \quad \text{and} \quad (-5)^2 = 25 \quad \text{hence} \quad \sqrt{25} = \pm 5$$

However, real problems necessitate calculating $\sqrt{-25}$. To defer this frequently encountered question, mathematicians (such as Euler who defined the symbol $i \triangleq \sqrt{-1}$) write

$$\sqrt{-25} = \sqrt{25} \sqrt{-1} = \pm 5 \sqrt{-1} = \pm 5i$$

Note: there are unresolved left/right order-of-operations conventions for exponents e.g.,

$$2^{3^2} = (2^3)^2 = 64 \quad \text{or} \quad 2^{3^2} = 2^{(3^2)} = 512$$

7. **Logarithms:** **Exponents** and **logarithms** are related. For example: $3^5 = 243$ and $\log_3 243 = 5$. A central issue of calculating logarithms is properly determining the **logarithm's base**.

Properties of exponents and logarithms

Addition	$x^{a+b} =$
Negation	$x^{-b} = 1/x^b$
Subtraction	$x^{a-b} = x^{a+^{-b}} = x^a x^{-b} = \frac{x^a}{x^b}$ Subtraction properties can be deduced from addition and negation properties
Exponentiation	$(x^a)^b = (x^b)^a = x^{(a*b)}$ Valid for some values of x, a, and b. Invalid for negative x or non-integer a and b , e.g., $[(-4)^2]^{\frac{1}{2}} \neq [(-4)^{\frac{1}{2}}]^2$. Also, parentheses avoid confusion e.g., $2^{3^2} = (2^3)^2 = 8^2 = 64$ or $2^{3^2} = 2^{(3^2)} = 2^9 = 512$?
Multiplication	$\log(a * b) =$
Exponentiation	$\log(a^n) =$
Division	$\log(a/b) = \log(a*b^{-1}) = \log(a) + \log(b^{-1}) = \log(a) + ^{-1}\log(b) = \log(a) - \log(b)$ The division properties of logarithms can be deduced from the negation, multiplication, and exponential properties of logarithms.

2.3.1 Order of operations

The mathematical convention for the order of operations is sometimes taught with the mnemonic **PEMDAS** (Parentheses Exponents Multiplication Division Addition Subtraction). PEMDAS is helpful for

many calculations and can be used in reverse to solve equations, but it fails to address *negation* and the left-right order of exponents. As mentioned earlier, there are different conventions used in textbooks, compilers, mathematical programs, and spreadsheets for the binding of the negative sign with a number. For example, is

$$-3^2 = (-3)^2 = +9? \quad \text{or} \quad -3^2 = (-1) * 3^2 = -9?$$

The left-right order of operations for exponents can also be interpreted in different ways, e.g.,

$$2^{3^2} = (2^3)^2 = 64 \quad \text{or} \quad 2^{3^2} = 2^{(3^2)} = 512?$$

2.3.2 Patterns in mathematics

There are many patterns in mathematics that are most recognizable with **three or more items**. This applies to the product rule for differentiation (see Section 2.8.6), the calculation of a determinate, and the transpose rule for matrices, e.g., $(ABC)^T = C^T B^T A^T$. For example, a simple, extensible way to clear parentheses is to start with the 1st term inside the first parentheses and multiply it by each successive term inside the second parentheses, then restart with the 2nd term inside the first parentheses, etc., e.g.,

$$(a + b + c)(d + e + f) = ad + ae + af + bd + be + bf + cd + ce + cf$$

It is unfortunate that many students learn the mnemonic **FOIL** to multiply parenthesized expressions containing **two** terms. The **FOIL** method is not an efficient way to multiply parenthesized expressions containing **three** or more terms.

2.4 Units

Units quantify the measurement of “stuff”. First adopted by France on December 10, 1799 and now used in all countries other than the United States,⁶ Liberia, and Myanmar, the **SI** system (International System of Units) is used to measure length (meters), mass (kilogram), force (Newton), temperature (Celsius), time (second), etc. Accurate conversions from one unit of measure to another (e.g., $\frac{\text{km}}{\text{hour}}$ to $\frac{\text{m}}{\text{sec}}$) are important as inaccuracies have led to many engineering failures. **NIST** (National Institute of Standards & Technology) defines and quantifies accurate conversion factors and physical constants.

2.5 Geometry

Geometry is the study and measurement of lines, curves, edges, surfaces, solids, etc. Geometry plays a central role in construction, farming, engineering, medicine, science, cooking, and in measuring nearly anything having distance, area, or volume.

In many K-12 educational systems, 2⁺ years are spent on 2D Euclidean geometry and trigonometry. The relatively new invention of **vectors** (Gibbs \approx 1900 AD) has significantly simplified 2D geometry and extended it to 3D geometry. The concepts, calculations, and applications of vector addition, dot-products, and cross-products are easy to teach, learn, and apply. Unfortunately, relatively few instructors are fully aware of the significant advantages of vectors in geometry.

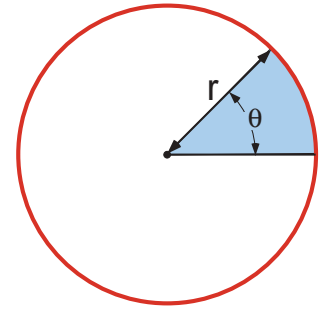
⁶It is ironic that the United States has not standardized on the SI system as it was the first country (in 1792) to use a monetary system with decimals and a base-10 number system. A study in Australia found that switching from British units to metric units freed an extra $\frac{1}{2}$ -year in science education.

2.6 Circles and their properties

The ratio of a circle's *circumference* to its *diameter* is a number equal to^a

$$\pi = 3.14159265358979323846264338327950288419716939937510582 \dots$$

π is called an “*irrational number*” because its value is not a whole number or fraction, nor does it terminate or repeat. It is chaotic, disorderly, and has no discernible pattern (π is as rational as the contests to memorize 67, 890⁺ digits of π).



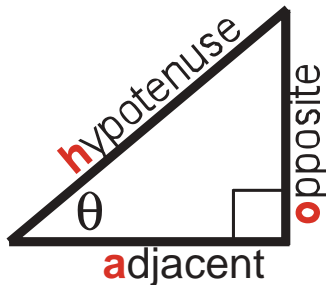
^aThe symbol π was introduced by Euler circa 1750, but the value $\pi \approx 3.14$ was known in Egypt around 3000 BC

The *arc-length* of a portion of the circle's periphery and the *area* of a wedge of the circle can be calculated in terms of the circle's *radius* r and the *angle* θ as⁷

Arc-length	=	θr	Area of wedge	=	$\frac{\theta}{2} r^2$
Circumference	=	$2\pi r$	Area of circle	=	πr^2

2.7 Triangles and the ratios of their sides

Triangles have a simple geometric shape and have been widely used in construction, surveying, and astronomy. The definitions of *sine*, *cosine*, and *tangent* as ratios of various sides of a right triangle are very old, dating back to before 140 BC when the Greek Hipparchus made sine, cosine, and tangent tables. More precise calculations of sine and cosine were made in 100 AD by the mathematician Ptolemy who calculated sines and cosines to six decimal places. A helpful *mnemonic* for memorizing the definitions of Sine, Cosine, and Tangent is *SohCahToa*.



$$\begin{aligned} \sin(\theta) &\triangleq \frac{\text{opposite}}{\text{hypotenuse}} \\ \cos(\theta) &\triangleq \frac{\text{adjacent}}{\text{hypotenuse}} \\ \tan(\theta) &\triangleq \frac{\text{opposite}}{\text{adjacent}} = \frac{\sin(\theta)}{\cos(\theta)} \end{aligned}$$

The mathematician Pythagoras of Samos (580-500 BC) was the first to *prove* an important property of a right triangle that had been widely used thousands of years earlier by the Babylonians. The Pythagorean Theorem relates the lengths of the sides of a right triangle, namely

$$\text{hypotenuse}^2 = \text{adjacent}^2 + \text{opposite}^2 \quad (1)$$

A second important relationship that follows directly from the Pythagorean Theorem and the definitions of $\sin(\theta)$ and $\cos(\theta)$ is⁸

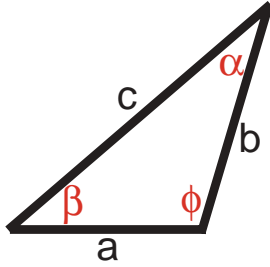
$$\sin^2(\theta) + \cos^2(\theta) \stackrel{(1)}{=} 1 \quad (2)$$

⁷An *angle* always involves two lines and is measured in radians or degrees. A radian is the ratio of the arc-length of a circle to the circle's radius. A degree is an archaic unit of measurement for an angle. Since there are approximately 360 days in a year, each degree represents one day of Earth's travel about the sun. The degree symbol looks like a small circle as a reminder that 360° measures the Earth's circular travel around the sun. Historically, the early Babylonian number system was based on 60, and the numbers 360 (days in a year) and 30 (days in a month) were sacred numbers.

⁸Numbers under an equals sign refer to equations numbered correspondingly.

2.7.1 Properties of sine, cosine, and tangent

There are many useful trigonometric formulas, including the **Law of Cosines** (Euclid of Alexandria 300 BC), the **Law of Sines** (Ptolemy of Alexandria Egypt 100 AD), and the **addition formula for sine** (Ptolemy 100 AD).^{9 10}



$$c^2 = a^2 + b^2 - 2ab \cos(\phi) \quad \text{Law of cosines} \quad (3)$$

$$\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\phi)}{c} \quad \text{Law of sines} \quad (4)$$

$$\sin(-\alpha) = \quad \quad \quad (5)$$

$$\cos(-\alpha) = \quad \quad \quad (6)$$

$$\sin(\alpha + \beta) = \quad \quad \quad \text{Addition formula for sine} \quad (7)$$

$$\cos(\alpha + \beta) = \quad \quad \quad \text{Addition formula for cosine} \quad (8)$$

$$\sin(x) = \sin(x + 2\pi n) \quad n = 1, 2, 3, \dots \quad \text{Sine is periodic} \quad (9)$$

$$\cos(x) = \cos(x + 2\pi n) \quad n = 1, 2, 3, \dots \quad \text{Cosine is periodic} \quad (10)$$

$$\sin(x) = \cos\left(x - \frac{\pi}{2}\right) = \cos\left(-x + \frac{\pi}{2}\right) \quad (11)$$

$$\cos(x) = \sin\left(x + \frac{\pi}{2}\right) = \sin\left(-x + \frac{\pi}{2}\right) \quad (12)$$

$$-\sin(x) = \sin(x \pm \pi n) \quad n=1, 3, 5, \dots \quad (13)$$

$$-\cos(x) = \cos(x \pm \pi n) \quad n=1, 3, 5, \dots \quad (14)$$

$$\sin(x) = 2 \sin\left(\frac{x}{2}\right) \cos\left(\frac{x}{2}\right) \quad \text{or} \quad \sin(2x) = 2 \sin(x) \cos(x) \quad (15)$$

$$\sin(x)^2 = \frac{1 - \cos(2x)}{2} \quad (16)$$

$$\cos(x)^2 = \frac{1 + \cos(2x)}{2} \quad (17)$$

$$\sin\left(\frac{x}{2}\right)^2 = \frac{1 - \cos(x)}{2} \quad \text{or} \quad \cos(x) = 1 - 2 \sin\left(\frac{x}{2}\right)^2 \quad (18)$$

$$\cos\left(\frac{x}{2}\right)^2 = \frac{1 + \cos(x)}{2} \quad \text{or} \quad \cos(x) = 2 \cos\left(\frac{x}{2}\right)^2 - 1 \quad (19)$$

$$\cos(b) - \cos(a) = 2 \sin\left(\frac{a+b}{2}\right) \sin\left(\frac{a-b}{2}\right) \quad \text{Useful for beat phenomenon analysis} \quad (20)$$

⁹Vectors are very useful in geometry and are a simple way to prove the **Law of Cosines**, **Law of Sines**, and **Addition formula for sine**, from which most other trigonometric formulas are derived (e.g., **Addition formula for cosine**, **half-angle formulas**, **double-angle formulas**, etc.).

¹⁰The trigonometric identities required to reverse-engineer equation (20) include $\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$, $\sin(x)^2 + \cos(x)^2 = 1$, and $\cos\left(\frac{x}{2}\right)^2 = \frac{1 + \cos(x)}{2}$.

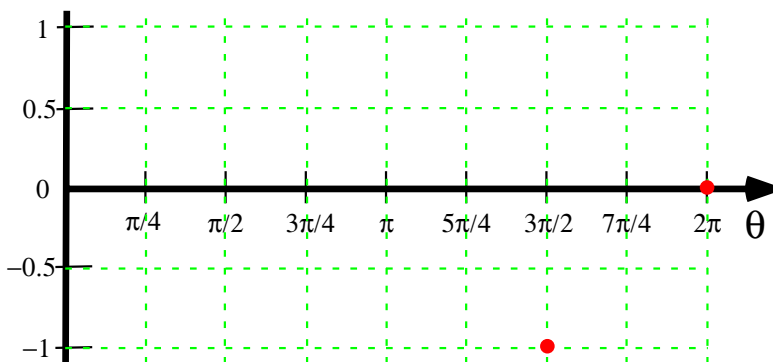
$$\cos(\omega_2 t + \phi_2) - \cos(\omega_1 t + \phi_1) \stackrel{(20)}{=} 2 \sin \left[\left(\frac{\omega_1 + \omega_2}{2} \right) t + \frac{\phi_1 + \phi_2}{2} \right] \sin \left[\left(\frac{\omega_1 - \omega_2}{2} \right) t + \frac{\phi_1 - \phi_2}{2} \right] \quad (21)$$

2.7.2 Sine, cosine, and tangent as functions

The functional character of sine, cosine, and tangent was discovered by Euler circa 1730. Thinking of $\sin(\theta)$ and $\cos(\theta)$ as **functions**, not just as the ratio of sides of a triangle, was a major advance for trigonometry. Complete the graph of $\sin(\theta)$ in Figure 2.1 and $\cos(\theta)$ in Figure 2.2 by sketching the appropriate curves.

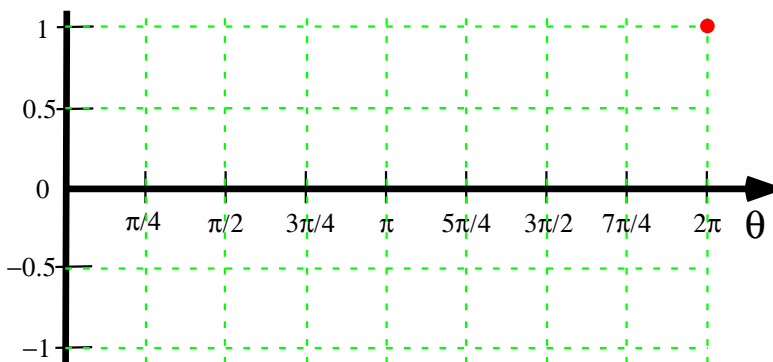
$$\sin(\theta) \triangleq \frac{\text{opposite}}{\text{hypotenuse}}$$

Figure 2.1: sin function



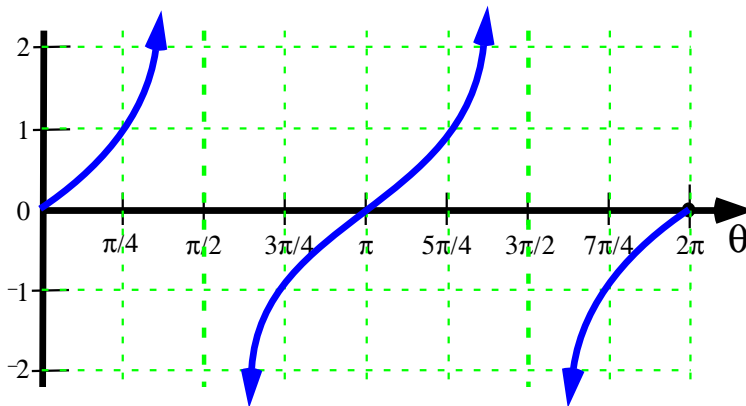
$$\cos(\theta) \triangleq \frac{\text{adjacent}}{\text{hypotenuse}}$$

Figure 2.2: cos function



$$\tan(\theta) \triangleq \frac{\text{opposite}}{\text{adjacent}} = \frac{\sin(\theta)}{\cos(\theta)}$$

Figure 2.3: tan function



2.7.3 The amplitude-phase formulas for sine and cosine

Two trigonometric identities that are particularly helpful in dynamic systems are the *amplitude-phase formulas for sine and cosine*.¹¹

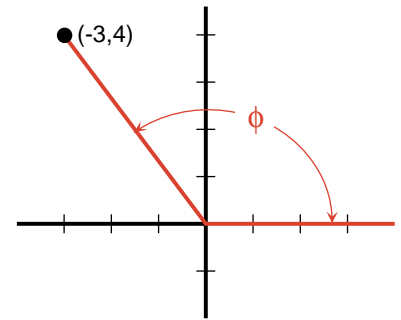
$$A \sin(x) + B \cos(x) = C \sin(x + \phi) \quad \text{where } C = +\sqrt{A^2 + B^2} \quad \text{and } \phi = \text{atan2}(B, A) \quad (22)$$

$$A \sin(x) + B \cos(x) = C \cos(x + \phi) \quad \text{where } C = +\sqrt{A^2 + B^2} \quad \text{and } \phi = \text{atan2}(-A, B) \quad (23)$$

2.7.4 The atan2 function

The *atan2* function is named because of its similarity to the arc-tangent function *atan* and because it takes two arguments. It differs from the *atan* function because the *atan2* function returns a value ϕ in the range $-\pi < \phi \leq \pi$ whereas the *atan* function returns $-\frac{\pi}{2} < \phi \leq \frac{\pi}{2}$. One way to determine the angle ϕ returned by the function $\phi = \text{atan2}(y, x)$ is:

- Draw a horizontal and vertical axis as shown on the right.
- Mark the point located at the designated y and x values.
For example, to calculate $\text{atan2}(4, -3)$ mark the point located at $y = 4$ and $x = -3$.
- Draw a line L connecting that point to the point at $(0, 0)$.
- Mark the angle ϕ between the positive x-axis and line L .
- Using trigonometry, calculate the value of ϕ , e.g., $\phi = 2.21$ rads.



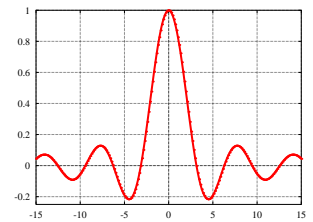
In computer programs such as Matlab, **M**otion**G**enesis, Java, C, and C++, the function $\text{atan2}(y, x)$ returns the radian measure of an angle $-\pi < \phi \leq \pi$ that satisfies both

$$\sin(\phi) = \frac{y}{+\sqrt{x^2 + y^2}} \quad \text{and} \quad \cos(\phi) = \frac{x}{+\sqrt{x^2 + y^2}} \quad \Rightarrow \quad \phi = \text{atan2}(y, x) \quad (24)$$

2.7.5 Optional*: The sinc function

The sinc function (also call the *sine cardinal* or *sampling function*) arises frequently in *Fourier transforms* and signal processing and is defined as

$$\text{sinc}(x) \triangleq \begin{cases} 1 & \text{for } x = 0 \\ \frac{\sin(x)}{x} & \text{for } x \neq 0 \end{cases}$$



2.8 Differentiation

2.8.1 Definition of a partial derivative of a scalar function

When a function f is regarded to depend on n independent scalar variables t_1, \dots, t_n , it is denoted¹² $f(t_1, \dots, t_n)$, and n quantities, called “first partial derivatives of f ”, can be formed. These quantities are denoted $\frac{\partial f}{\partial t_i}$ and are defined as

$$\frac{\partial f}{\partial t_i} \triangleq \lim_{h \rightarrow 0} \frac{f(t_1, \dots, t_i + h, \dots, t_n) - f(t_1, \dots, t_i, \dots, t_n)}{h} \quad (i = 1, \dots, n) \quad (25)$$

¹¹The amplitude-phase formulas [equations (22) and (23)] are used extensively in vibration analysis and are helpful for completing Homework 1.5. These general formulas require *atan2* because A and B may be **positive**, **negative**, or **zero**.

¹²The notation $f(t_1, \dots, t_n)$ was invented by Euler circa 1730.

Because $\frac{\partial f}{\partial t_i}$ is defined as a limit, it cannot be regarded as a ratio, meaning one cannot cancel the ∂t_i in the denominator by multiplying through by ∂t_i . In other words ∂t_i is not an entity in its own right.

2.8.2 Definition of an ordinary derivative of a scalar function

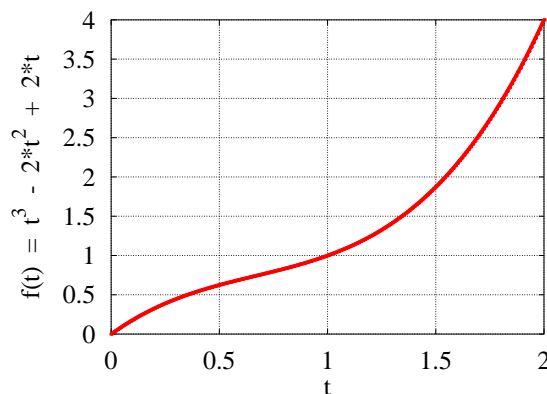
When a function f is regarded as a function of only *one* single scalar variable t , then the definition in equation (25) reduces to that of the “ordinary derivative of f with respect to t ”.¹³ Frequently, other symbols are used in place of $\frac{\partial f}{\partial t}$ to denote the ordinary derivative of f with respect to t . e.g.,^{14 15}

$$f' = f'(t) = \dot{f} = \dot{f}(t) = \frac{df}{dt} = \frac{d[f(t)]}{dt} = \frac{\partial f}{\partial t} \triangleq \lim_{h \rightarrow 0} \frac{f(t+h) - f(t)}{h} \quad (26)$$

Geometrical interpretation of an ordinary derivative

When f is a function of a single independent variable t , the (first) derivative of f with respect to t is the **slope**^a of $f(t)$. The derivative of $f'(t)$, also called the “second derivative of $f(t)$ with respect to t ”, is denoted $f''(t)$ and is the **curvature** of $f(t)$.

^aIt is easy to fit a curve that passes through a few points with specific slopes, e.g., with tangents to the curve at $t=0$, $t=1$, and $t=2$. This is the fundamental idea behind numerical integration.



2.8.3 Definition of the differential of a scalar function

When f is regarded as a function of n independent scalar variables t_1, \dots, t_n , one may define a quantity df called the **differential of the function f** in terms of dt_1, \dots, dt_n (**differentials of the independent variables** t_1, \dots, t_n). These “independent differentials” are defined to be arbitrary (usually small) quantities that have the same dimension of t_1, \dots, t_n . With dt_1, \dots, dt_n in hand, df is defined as

$$df \triangleq \frac{\partial f}{\partial t_1} * dt_1 + \frac{\partial f}{\partial t_2} * dt_2 + \dots + \frac{\partial f}{\partial t_n} * dt_n \quad (27)$$

When f is regarded as a function of a **single** scalar variable t , equation (27) reduces to

$$df \stackrel{(27)}{=} \frac{\partial f}{\partial t} * dt \quad (28)$$

¹³Synonyms for ordinary are “plain” and “boring” because f is a function of a *single* variable, whereas a “hot and spicy” partial derivative is usually a function of **two or more variables**.

¹⁴**History:** The modern differential notion $\frac{df}{dt}$ was introduced by Gottfried Leibniz in 1675. The dot-notation \dot{f} was introduced by Newton in his “**method of fluxions**” around 1675 and relates to idea of flux (time-rates of change) of “**fluents**” (now called **variables**). The prime notation was introduced by Lagrange in 1797 in his *Théorie des fonctions analytiques*. Lagrange called $f'(t)$ the “derived function” of $f(t)$, from which the modern term **derivative** comes [22, pgs. 95-97].

Although Newton and Leibniz shared the discovery of **calculus**, their initially cordial relationship turned contentious - with Newton and Leibniz and their respective supporters alleging plagiarism and undermining each other’s credibility. Ironically, the recluse Newton died at 80-years old a national hero of England with a state funeral of the highest honors. The more sociable Leibniz’s died at 70-years old, almost completely forgotten, with a funeral attended by only his secretary. Newton’s daunting reputation intimidated British mathematicians. England did not produce a single first-rate mathematician for over a century - and England’s next contributors were in algebra (not analysis). Undaunted and unintimidated by their English neighbors, the rest of Europe, lead by the Bernoulli family, Leonard Euler, and many others, quickly expanded analytical analysis through differential equations, the calculus of variations, etc.

¹⁵The chronological development of mathematics can occur in an interesting order. For example, the derivative was first **used** (Fermat and Descartes, 1637), then **discovered** (Newton and Leibniz, 1669-1684), then **explored** and **developed** (Taylor, Euler, Maclaurin, Lagrange, 1755-1797) and finally **defined** (Cauchy and Weierstrass, 1823-1861) [15].

One is free to divide both sides of equations (27) or (28) by any non-zero quantity, including a differential. Dividing both sides of equation (28) by dt gives rise to the **ratio** of df to dt , that is

$$\frac{df}{dt} \underset{(28)}{=} \frac{\partial f}{\partial t} \quad (29)$$

Hence, when f is a function of a **single** independent scalar variable, the symbol $\frac{df}{dt}$ can mean both a **ratio** of the differential df to the differential dt and as a **limit** (or **ordinary derivative**) in the sense of equation (26). This “overloading” of the symbol $\frac{df}{dt}$ can be both useful and confusing.

2.8.4 Definition of the total derivative of a scalar function

At times, a function f can be regarded as either depending on a single scalar quantity t , or regarded as a function of $n + 1$ independent scalar quantities x_1, \dots, x_n and t , where x_1, \dots, x_n are themselves functions of t . When one regards f as a function of x_1, \dots, x_n and t , f is denoted $f(x_1(t), \dots, x_n(t), t)$, and the ordinary derivative of f with respect to t is called the **total derivative** of f with respect to t and can be calculated as

$$\frac{df}{dt} = \frac{\partial f}{\partial x_1} * \frac{dx_1}{dt} + \frac{\partial f}{\partial x_2} * \frac{dx_2}{dt} + \dots + \frac{\partial f}{\partial x_n} * \frac{dx_n}{dt} + \frac{\partial f}{\partial t} \quad (30)$$

2.8.5 Short table of derivatives frequently encountered in engineering

Function	Derivative of $F(t)$
$F(t) = t^n$	$\frac{\partial F}{\partial t} = n * t^{n-1}$ (n = constant)
$F(t) = \ln(t)$	$\frac{\partial F}{\partial t} =$
$F(t) = e^t$	$\frac{\partial F}{\partial t} =$ Very important for differential equations
$F(t) = \sin(t)$	$\frac{\partial F}{\partial t} =$
$F(t) = \cos(t)$	$\frac{\partial F}{\partial t} =$
$F(t) = \tan(t)$	$\frac{\partial F}{\partial t} = \frac{1}{\cos^2(t)}$
$F(t) = \int_{x=t_0}^t f(x) dx$	$\frac{\partial F}{\partial t} = f(t)$ (Fundamental Theorem of Calculus)
$F(t) = \int_{s=g(t)}^{h(t)} f(s, t) ds$	$\frac{\partial F}{\partial t} = \int_{s=g(t)}^{h(t)} \frac{\partial f(s, t)}{\partial t} ds - f[s=g(t), t] \frac{d[g(t)]}{dt} + f[s=h(t), t] \frac{d[h(t)]}{dt}$

2.8.6 Product rule for derivatives

Many calculus books introduce the “**bad**” product rule for differentiation $\frac{d(u * v)}{dt} = u * \frac{dv}{dt} + v * \frac{du}{dt}$. This is unfortunate as this product rule does not work if u and v are matrices, vectors, dyadics, etc. Additionally, it is difficult to use this product rule to differentiate the product of three or more scalar quantities, e.g., $u * v * w$. A simple, efficient, and extensible “**good**” **product rule for differentiation**, that works for matrices, vectors, dyadics, etc., is¹⁶

$$\frac{\partial(u * v * w)}{\partial t} = \frac{\partial u}{\partial t} * v * w + u * \frac{\partial v}{\partial t} * w + u * v * \frac{\partial w}{\partial t} \quad (31)$$

¹⁶The product rule for derivatives is useful for completing Homework 1.7.

Product rule example: $\frac{\partial [t^2 * \sin(t) * e^t]}{\partial t} = \text{[yellow box]} + t^2 \cos(t) e^t + \text{[yellow box]}$

2.8.7 Quotient rule for derivatives using exponents and the product rule

Since the quotient $\frac{u}{v}$ is equivalent to $u v^{-1}$, the partial derivative of $\frac{u}{v}$ with respect to t is implemented with the product rule and exponents (without memorizing special *quotient-rule* formulas) as

$$\frac{\partial}{\partial t} \left(\frac{u}{v} \right) = \frac{\partial u}{\partial t} v^{-1} - u v^{-2} \frac{\partial v}{\partial t} \quad (32)$$

2.8.8 Chain rule for derivatives

When the variable x depends on the variable t , the partial derivative of the function $f(x)$ with respect to t can be written via the *chain rule for differentiation* as shown in equation (33).

$$\frac{\partial f(x)}{\partial t} = \frac{\partial f(x)}{\partial x} \frac{\partial x}{\partial t} \quad (33)$$

2.8.9 Implicit differentiation

Implicit differentiation can be a useful tool for efficiently calculating derivatives. For example, the following **nonlinear** algebraic equation relates a dependent variable y to an independent variable t .

$$y^2 + \sin(y) = \cos(t)$$

In general, it is difficult to solve a nonlinear equation to find y explicitly in terms of t . However, implicit differentiation calculates $\frac{dy}{dt}$ **without** first solving for y , e.g., differentiating the previous equation gives

$$2y \frac{dy}{dt} + \cos(y) \frac{dy}{dt} = -\sin(t) \quad \Rightarrow \quad \frac{dy}{dt} = \frac{-\sin(t)}{2y + \cos(y)}$$

The use of implicit differentiation in conjunction with *natural logarithms* is useful for calculating the ordinary time-derivative of $y = c^t$ (c is a constant and t is time), as shown below.¹⁷

$$y = c^t \quad \Rightarrow \quad \ln(y) = t \ln(c) \quad \Rightarrow \quad d[\ln(y)] = \ln(c) dt \quad \Rightarrow \quad \frac{1}{y} dy = \ln(c) dt$$

Solving for the ratio of dy to dt (which is equal to the ordinary time-derivative of y), yields

$$\frac{dy}{dt} = \ln(c) y = \ln(c) c^t$$

Note: When $c = e = 2.718281828$, $\frac{dy}{dt} = y$, which plays a **central role** in ordinary differential equations.

2.8.10 Differentiation with the symbolic manipulator MotionGenesis

Symbolic manipulators are useful for calculating *partial derivatives* and *ordinary time-derivatives*. For example, typing the following input lines at the MotionGenesis prompt, produces the results below.

Note: Output lines are marked with ->.

```
(1) Variable x, y
(2) z = y*cos(x) + 2*x^2*sin(y)
-> (3) z = y*cos(x) + 2*x^2*sin(y)

(4) partialDerivativeOfZwithRespectToY = D( z, y )
-> (5) partialDerivativeOfZwithRespectToY = cos(x) + 2*x^2*cos(y)

(6) partialDerivativeOfZwithRespectToX = D( z, x )
```

¹⁷Implicit differentiation can be done with derivatives (shown in the first example in Section 2.8.9) or differentials (shown in the second example in Section 2.8.9).

```

-> (7) partialDerivativeOfZWithRespectToX = 4*x*sin(y) - y*sin(x)

(8) Variable s' % Declares s as a variable and s' as it's ordinary time-derivative
(9) funct = log(s) + s*exp(s)
-> (10) funct = log(s) + s*exp(s)

(11) ordinaryTimeDerivativeOfFunct = Dt( funct )
-> (12) ordinaryTimeDerivativeOfFunct = (1/s+exp(s)+s*exp(s))*s'

```

2.9 Integration and a short table of integrals

An *integral* can be regarded as a sum or as an anti-derivative. A useful interpretation of an integral is the *area* under a curve. The following table contains integrals commonly found in engineering analysis.¹⁸

Function	Integral of $F(t)$
$F(t) = t^n$	$\int F(t) dt = \frac{t^{n+1}}{n+1} + C$ (n is a number other than -1)
$F(t) = t^{-1}$	$\int F(t) dt = \ln(t) + C$
$F(t) = e^t$	$\int F(t) dt = e^t + C$
$F(t) = \sin(t)$	$\int F(t) dt = -\cos(t) + C$
$F(t) = \cos(t)$	$\int F(t) dt = \sin(t) + C$

2.10 Solutions of *linear algebraic equations*

It is relatively easy to solve a single, uncoupled, *linear algebraic equation*, e.g., solving for x in

$$3x + 9 \sin(t) - 12 = 0$$

Solving two *coupled linear algebraic equations* for y and z is a little more difficult, e.g.,

$$\begin{aligned} 3y + 2z + 9 \sin(t) - 12 &= 0 \\ 2y + 4z + 5 \cos(t) - 11 &= 0 \end{aligned}$$

Solving four *coupled linear algebraic equations* for x_1, x_2, x_3, x_4 is more difficult, e.g.,

$$\begin{aligned} 3x_1 + 2x_2 + 2x_3 + 3x_4 &= 9 \sin(t) \\ 2x_1 + 4x_2 + 2x_3 + 3x_4 &= 5 \cos(t) \\ 4x_1 + 5x_2 + 6x_3 + 7x_4 &= 11 \\ 9x_1 + 8x_2 + 7x_3 + 6x_4 &= 15 \end{aligned}$$

2.10.1 Solutions of coupled *linear algebraic equations* with **MotionGenesis** (symbolic)

MotionGenesis¹⁹ **symbolically** solves the linear algebraic equations in Section 2.10 with the commands:

```

%-----
Variable x
Equation = 3*x + 9*sin(t) - 12
Solve( Equation, x )

```

¹⁸The website www.Mathematica.com is a valuable resource for calculating integrals.

¹⁹**MotionGenesis** is a symbolic manipulator (like Mathematica and Maple) for **engineers** and is a popular tool for generating and solving linear and nonlinear algebraic and differential equations and writing C, Matlab, and Fortran code.

```

%-----
Variable y, z
Zero[1] = 3*y + 2*z + 9*sin(t) - 12
Zero[2] = 2*y + 4*z + 5*cos(t) - 11
Solve( Zero, y, z )
%-----
Variable x{1:4}
Eqn[1] = 3*x1 + 2*x2 + 2*x3 + 3*x4 - 9*sin(t)
Eqn[2] = 2*x1 + 4*x2 + 2*x3 + 3*x4 - 5*cos(t)
Eqn[3] = 4*x1 + 5*x2 + 6*x3 + 7*x4 - 11
Eqn[4] = 9*x1 + 8*x2 + 7*x3 + 6*x4 - 15
Solve( Eqn, x1, x2, x3, x4 )
%-----
Save SolveLinearEquations.all

```

2.10.2 Solutions of coupled *linear* algebraic equations with Matlab (numeric)

Matlab **numerically** solves the linear algebraic equations in Section 2.10 by assigning a numerical value to t and typing the commands:

```

t = 0.2;
%-----
Coef(1,1) = 3;   Rhs(1,1) = -(9*sin(t) - 12);
SolutionToAxEqualsB = Coef \ Rhs;
x = SolutionToAxEqualsB(1)
%-----
Coef(1,1) = 3;   Coef(1,2) = 2;   Rhs(1,1) = -(9*sin(t) - 12);
Coef(2,1) = 2;   Coef(2,2) = 4;   Rhs(2,1) = -(5*cos(t) - 11);
SolutionToAxEqualsB = Coef \ Rhs;
y = SolutionToAxEqualsB(1)
z = SolutionToAxEqualsB(2)
%-----
Coef(1,1) = 3;   Coef(1,2) = 2;   Coef(1,3) = 2;   Coef(1,4) = 3;   Rhs(1,1) = 9*sin(t);
Coef(2,1) = 2;   Coef(2,2) = 4;   Coef(2,3) = 2;   Coef(2,4) = 3;   Rhs(2,1) = 5*cos(t);
Coef(3,1) = 4;   Coef(3,2) = 5;   Coef(3,3) = 6;   Coef(3,4) = 7;   Rhs(3,1) = 11;
Coef(4,1) = 9;   Coef(4,2) = 8;   Coef(4,3) = 7;   Coef(4,4) = 6;   Rhs(4,1) = 15;
SolutionToAxEqualsB = Coef \ Rhs;
x1 = SolutionToAxEqualsB(1)
x2 = SolutionToAxEqualsB(2)
x3 = SolutionToAxEqualsB(3)
x4 = SolutionToAxEqualsB(4)

```

2.11 Solutions of *polynomial* equations (roots)

Polynomial equations are a special class of nonlinear algebraic equations. A special polynomial equation is the *quadratic equation*, which is a polynomial equation of degree **2**. Shown below is a quadratic equation in x and its **2 roots** (solutions).

Quadratic equation

$$a x^2 + b x + c = 0$$

Solution to quadratic equation

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Two other polynomial equations with “closed-form solutions” are the *cubic* and *quartic* equations

$$x^3 + c_2 x^2 + c_1 x + c_0 = 0 \quad \text{and} \quad x^4 + c_3 x^3 + c_2 x^2 + c_1 x + c_0 = 0$$

In 1824, Abel proved that no general closed-form solution for 5^{th} -order (or higher) polynomials exist. It is known, through the *fundamental theorem of algebra*, that any polynomial of degree n with complex

coefficients has n complex roots.²⁰

Roots of a quadratic equation with MotionGenesis (symbolic)

The following shows how MotionGenesis calculates the roots of $ax^2 + bx + c = 0$.

```
(1) %-----
(2) % Example 1: GetQuadraticRoots (roots of quadratic equation)
(3) %-----
(4) Constant a, b, c
(5) Variable x
(6) rootsA = GetQuadraticRoots( a*x^2 + b*x + c, x )
-> (7) rootsA[1] = -0.5*(b-sqrt(b^2-4*a*c))/a
-> (8) rootsA[2] = -0.5*(b+sqrt(b^2-4*a*c))/a

(9) %-----
(10) % Example 2: GetQuadraticRoots (roots of quadratic equation)
(11) %-----
(12) rootsB = GetQuadraticRoots( [a; b; c] )
-> (13) rootsB[1] = -0.5*(b-sqrt(b^2-4*a*c))/a
-> (14) rootsB[2] = -0.5*(b+sqrt(b^2-4*a*c))/a
```

Roots of a 5th-order polynomial with MotionGenesis and Matlab

The following codes shows how MotionGenesis and Matlab calculate roots of a 5th-order polynomial equation.

$$p^5 + 2*p^4 + 3*p^3 + 5*p^2 + 9*p + 17 = 0$$

Roots of a polynomial equation of degree 5 with MotionGenesis

```
(1) %-----
(2) % Example 1: GetPolynomialRoots (roots of 5th-order polynomial)
(3) %-----
(4) SetImaginary( i )
(5) Variable p
(6) rootsA = GetPolynomialRoots( p^5 + 2*p^4 + 3*p^3 + 5*p^2 + 9*p + 17, p, 5 )
-> (7) rootsA = [-1.857621; -0.9475112 - 1.507048*i; -0.9475112 + 1.507048*i;
0.8763218 - 1.455989*i; 0.8763218 + 1.455989*i]

(8) %-----
(9) % Example 2: GetPolynomialRoots (roots of 5th-order polynomial)
(10) %-----
(11) rootsB = GetPolynomialRoots( [1, 2, 3, 5, 9, 17] )
-> (12) rootsB = [-1.857621, -0.9475112 - 1.507048*i, -0.9475112 + 1.507048*i,
0.8763218 - 1.455989*i, 0.8763218 + 1.455989*i]
```

Roots of a polynomial equation of degree 5 with Matlab

```
>> polynomial = [1, 2, 3, 5, 9, 17];
>> p = roots( polynomial )
```

```
p =
    0.8763 + 1.4560i
    0.8763 - 1.4560i
   -1.8576
   -0.9475 + 1.5070i
   -0.9475 - 1.5070i
```

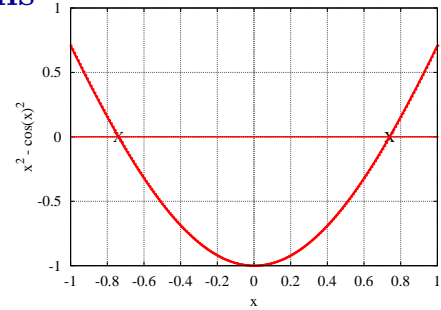
²⁰The proof of the *fundamental theorem of algebra* is difficult and was presented with various rigor between 1608 and 1981 by great mathematicians including, Rothe(1608) Girard (1629), Leibniz (1702), Bernoulli (1742), d'Alembert (1746), Euler (1749), Lagrange (1772), Laplace (1795), Gauss (1799), Argand (**1806**), Gauss (again in 1816 and 1849), Cauchy (1821), Weierstrauss (1891), Hellmuth Kneser (1940), and his son Martin Kneser (**1981**).

2.12 Solutions of *nonlinear* algebraic equations

One way to find the solution to the *nonlinear algebraic equation*

$$x^2 - \cos^2(x) = 0$$

is to graph the function $x^2 - \cos^2(x)$ vs. x and identify the values of x that make the function equal to 0. The graph shows this function is **nonlinear** (i.e., it is **not a line**) and has two solutions, namely $x = 0.7391$ and $x = -0.7391$.



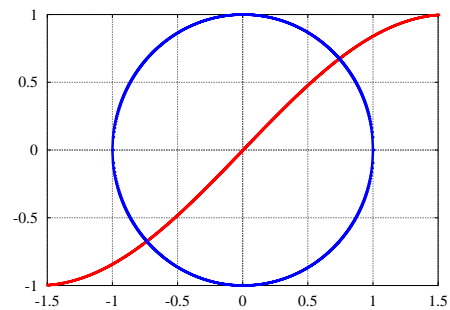
Another way to solve a nonlinear equation is to use a computer program such as **MotionGenesis** or Matlab. Most algorithms start with a guess and iterate towards a solution (usually the solution closest to the guess). For example, the following **MotionGenesis** commands produce the solution $x = 0.7391$.

```
Variable x
SolveNonlinear( x^2 - cos(x)^2, x=2 )    % x=2 is a guess to a solution
```

2.12.1 Solutions of *coupled nonlinear* algebraic equations with **MotionGenesis**

Shown to the right is a coupled set of algebraic equations that is **nonlinear**^a in x and y (the unit circle and sine curve are **not lines**). These two curves intersect at two locations (there are two solutions to these equations), namely $x = 0.7391$, $y = 0.6736$ and $x = -0.7391$, $y = -0.6736$. In general, it is difficult to determine the **number of solutions** to nonlinear algebraic equations, and the solution process usually requires a numerical algorithm that starts with a guess and then iterates towards a solution.

$$\begin{aligned} x^2 + y^2 &= 1 \\ y &= \sin(x) \end{aligned}$$



^aNonlinear equations with one or two unknowns can be solved by **trial and error** or **graphing**. In general, **Newton-Rhapson** techniques are used to solve sets of nonlinear equations.

For example, the following **MotionGenesis** commands produce the solution $x = -0.7391$, $y = -0.6736$.

```
Variable x, y                                % Unknowns
Zero[1] = x^2 + y^2 - 1                      % x^2 + y^2 = 1    (unit circle)
Zero[2] = y - sin(x)                         % y = sin(x)      (sine wave)
SolveNonlinear( Zero, x=1.5, y=0 )          % x=1.5, y=0 is a guess to a solution
```

2.12.2 Solutions of *coupled nonlinear* algebraic equations with Matlab

Matlab is another popular tool for solving nonlinear equations. To produce a Matlab solution for finding the intersection of a circle and a sine wave:

- Use a **text editor** to create the file `NonlinearSolveCircleSine.m` (as shown below)
- Invoke Matlab and make sure `NonlinearSolveCircleSine.m` is in the current working directory
- Type `NonlinearSolveCircleSine` at the Matlab prompt
- Note: The Matlab nonlinear solver `fsolve` requires the **optimization** toolbox.

```
%-----
%   File: NonlinearSolveCircleSine.m
%   Purpose: Solving a set of nonlinear equations with Matlab
%   Note: Requires Matlab's optimization toolbox
%-----
function solutionToNonlinearEquations = NonlinearSolveCircleSine
initialGuess = [ 2, 0 ];
solveOptions = optimset('fsolve');
```

```
solutionToNonlinearEquations = fsolve( @CalculateFunctionEvaluatedAtX, initialGuess, solveOptions );
```

```
%=====
function fx = CalculateFunctionEvaluatedAtX( X )
x = X(1);    y = X(2);
fx(1) = x^2 + y^2 - 1;    % x^2 + y^2 = 1    (unit circle)
fx(2) = y - sin(x);      % y = sin(x)      (sine wave)
```

2.13 Numerical solution of nonlinear 2nd-order ODEs

The figure to the right shows a 1 m long pendulum swinging on Earth's surface. The pendulum's motion is governed by the 2nd-order ODE

$$\ddot{\theta} = -9.8 \sin(\theta)$$

One way to solve this ODE is with computer *numerical integration*.

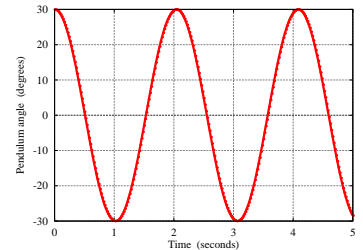
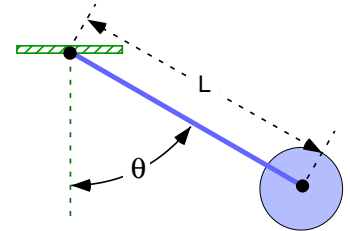
The **MotionGenesis** solution to this ODE is shown below.

The output data file `ClassicParticlePendulum.1` has two columns.

The first column records time from 0 to 5 sec in increments of 0.02 sec.

The second column records the associated value of θ , accurate to 1×10^{-8} .

```
Variable theta''
theta'' = -9.8*sin(theta)
%-----
Input  tFinal=5, integStp=0.02, absErr=1.0E-08, relErr=1.0E-08
Input  theta=30 deg, theta'=0
Output t, theta deg
%-----
Code ODE() ClassicParticlePendulum
```

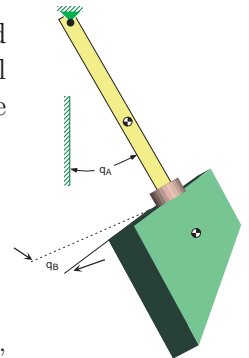


2.14 Solution of coupled nonlinear 2nd-order ODEs

Computers and software programs such as **MotionGenesis**, Matlab, Mathematica, and Maple have revolutionized the **numerical solution** of partial and ordinary differential equations.^a The following sections show how to use **MotionGenesis** and Matlab to solve the ordinary differential equations governing the motion of the system shown to the right.

$$\ddot{q}_A = \frac{2 [508.89 \sin(q_A) - \sin(q_B) \cos(q_B) \dot{q}_A \dot{q}_B]}{-21.556 + \sin(q_B)^2}$$

$$\ddot{q}_B = -\sin(q_B) \cos(q_B) \dot{q}_A^2$$



^aThere are important reasons to study **analytical solutions** of differential equations. For example, in control system design, the stability of a differential equation can be more important than its solution.

2.14.1 Solution of ordinary differential equations with **MotionGenesis**

MotionGenesis generates and solves linear and nonlinear differential equations and produces fast, efficient, distributable C, Fortran, and Matlab codes that can be used with Simulink.²¹ The following **MotionGenesis**

²¹Compiled C and Fortran codes are **hundreds of times faster** than interpreted codes such as Matlab. This difference is significant for embedded systems that require real-time operation or when compiled code requires more than a minute to execute (which means the interpreted code requires several hours). **MotionGenesis** symbolically optimizes expressions and sophisticated compilers optimize code for a specific operating system and microprocessor. To create C, Matlab, or Fortran code, replace the last line in `BabybootODE.a1` with: `Code ODE babyboot.c` or `Code ODE babyboot.m` or `Code ODE babyboot.f`

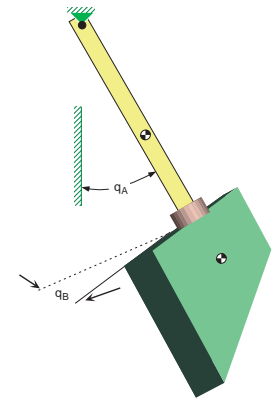
commands solve the differential equations in Section 2.14.²²

The output data file `solveBabybootODE.1` has three columns.

The first column records time from 0 to 10 sec in increments of 0.02 sec.

The second and third columns record associated values of q_A and q_B (in degrees).

```
Variable qA'', qB'' % Angles and their first and second time-derivatives
%-----
qA'' = 2*( 508.89*sin(qA) - sin(qB)*cos(qB)*qA'*qB' ) / (-21.556 + sin(qB)^2)
qB'' = -sin(qB)*cos(qB)*qA'^2
%-----
Input tFinal=10 sec, integStp=0.02 sec, absErr=1.0E-07, relErr=1.0E-07
Input qA=90 deg, qB=1.0 deg, qA'=0.0 rad/sec, qB'=0.0 rad/sec
Output t sec, qA degrees, qB degrees
%-----
Code ODE() solveBabybootODE
```



2.14.2 Solution of ordinary differential equations with Matlab

The Matlab solution for the differential equations in Section 2.14 has two functions. The first function is the main routine that drives the numerical integrator. The second function contains the differential equations in first-order form. Since the solution is sensitive to numerical inaccuracies, Matlab's default integration error parameters `RelTol` and `Abstol` were modified as shown. To use Matlab:

- Use a text editor to create the file `BabybootODE.m`
- Invoke Matlab and make sure `BabybootODE.m` is in the current working directory
- Type `BabybootODE` at the Matlab prompt

```
%-----
% File: BabybootODE.m (solving differential equations with Matlab)
%-----
function BabybootODE
degreesToRadians = pi/180;
initialState = [ 90*degreesToRadians 1.0*degreesToRadians 0 0 ];
timeInterval = linspace( 0, 10, 1000 );
odeOptions = odeset( 'RelTol', 1.0e-7, 'Abstol', [1.0E-8, 1.0E-8, 1.0E-8, 1.0E-8] );
[time,stateMatrix] = ode45( @odefunction, timeInterval, initialState, odeOptions );
qB = stateMatrix(:,2);
plot( time, qB/degreesToRadians, 'r-' )
xlabel( ' Time (seconds) ' );
ylabel( ' Plate angle (degrees) ' );

function timeDerivativeOfState = odefunction( t, state )
qA = state(1); % Pendulum angle
qB = state(2); % Plate angle
qAp = state(3); % qA', time derivative of the pendulum angle
qBp = state(4); % qB', time derivative of the plate angle
qApp = 2*( 508.89*sin(qA) - sin(qB)*cos(qB)*qAp*qBp ) / (-21.556 + sin(qB)^2);
qBpp = -sin(qB)*cos(qB)*qAp^2;
timeDerivativeOfState = [qAp; qBp; qApp; qBpp];
```

2.15 Optional*: Proofs

Trigonometric formulas can be tedious or difficult to prove. For example, a proof of the law of cosines can be found in the book *Engineering Mechanics OnLine*, by Thomas R. Kane and David A. Levinson, 1999. Several trigonometric proofs are shown below.

²²MotionGenesis commands can be typed at the MotionGenesis prompt or run from a file as follows.

- Use a **text editor**, e.g., NotePad (Windows), SimpleText (Macintosh), Pico, Emacs, or vi (Unix), to create the file `BabybootODE.al` (as shown in Section 2.14.1)
- Copy the file to the MotionGenesis folder, invoke MotionGenesis, and type `run BabybootODE.al` at the MotionGenesis prompt. Or, in Microsoft Windows, drag and drop the file onto the MotionGenesis icon

2.15.1 Proof of the addition formula for the sine function

One way to prove equation (7) and equation (8) is to note that Euler's formula provides

$$e^{i(\alpha+\beta)} = \cos(\alpha + \beta) + i \sin(\alpha + \beta) \quad (34)$$

A second way to write $e^{i(\alpha+\beta)}$ is to use the addition property of exponents, namely

$$\begin{aligned} e^{i(\alpha+\beta)} &= e^{i\alpha} * e^{i\beta} = (\cos \alpha + i \sin \alpha) * (\cos \beta + i \sin \beta) \\ &= \cos(\alpha) \cos(\beta) - \sin(\alpha) \sin(\beta) + i [\sin(\alpha) \cos(\beta) + \sin(\beta) \cos(\alpha)] \end{aligned} \quad (35)$$

Equating the real parts of the right hand-sides of equations (34) and (35) leads directly to equation (8). Similarly, equating the imaginary parts of the equations (34) and (35) leads directly to equation (7).

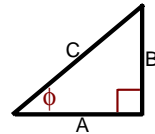
2.15.2 Geometrical proof of the amplitude-phase trigonometric identity

Multiplying and dividing the left-hand side of equation (22) by $+\sqrt{A^2 + B^2}$ leads to

$$A \sin(x) + B \cos(x) = +\sqrt{A^2 + B^2} \left[\frac{A}{+\sqrt{A^2 + B^2}} \sin(x) + \frac{B}{+\sqrt{A^2 + B^2}} \cos(x) \right] \quad (36)$$

The geometry of a right triangle is used to form expressions for C , $\cos(\phi)$, and $\sin(\phi)$.

$$C = +\sqrt{A^2 + B^2} \quad \cos(\phi) = \frac{A}{+\sqrt{A^2 + B^2}} \quad \sin(\phi) = \frac{B}{+\sqrt{A^2 + B^2}}$$



With the expressions for C , $\cos(\phi)$, and $\sin(\phi)$ in hand, equation (36) can be rewritten as

$$A \sin(x) + B \cos(x) \stackrel{(36)}{=} C [\cos(\phi) \sin(x) + \sin(\phi) \cos(x)] \quad (37)$$

In view of equation (7), it is possible to rewrite the right-hand side of equation (37) as

$$A \sin(x) + B \cos(x) \stackrel{(7,37)}{=} C \sin(x + \phi) \quad (38)$$

2.15.3 Trigonometric proof of the amplitude-phase trigonometric identity

An alternate mathematical proof of equation (22) begins with

$$C \cos(x + \phi) \stackrel{?}{=} A \sin(x) + B \cos(x) \quad (39)$$

$$C [\cos(x) \cos(\phi) - \sin(x) \sin(\phi)] \stackrel{(39)}{=} A \sin(x) + B \cos(x) \quad (40)$$

$$[C \cos(\phi) - B] \cos(x) - [C \sin(\phi) + A] \sin(x) \stackrel{(40)}{=} 0 \quad (41)$$

Since x may be assigned any value and $\cos(x)$ and $\sin(x)$ are linearly independent functions²³, the coefficients of both $\cos(x)$ and $\sin(x)$ must be zero. Thus,

$$C \cos(\phi) - B \stackrel{(41)}{=} 0 \quad \Rightarrow \quad C \cos(\phi) = B \quad (42)$$

$$C \sin(\phi) + A \stackrel{(41)}{=} 0 \quad \Rightarrow \quad C \sin(\phi) = -A \quad (43)$$

Equations (42) and (43) are two **nonlinear** algebraic equations in two unknowns (C and ϕ).

$$C^2 (\sin^2 \phi + \cos^2 \phi) \stackrel{(42,43)}{=} A^2 + B^2 \quad \Rightarrow \quad C \stackrel{(2)}{=} +\sqrt{A^2 + B^2}$$

$$\cos(\phi) \stackrel{(42)}{=} \frac{B}{C} \quad \text{and} \quad \sin(\phi) \stackrel{(43)}{=} \frac{-A}{C} \quad \Rightarrow \quad \phi \stackrel{(24)}{=} \text{atan2}(-A, B)$$

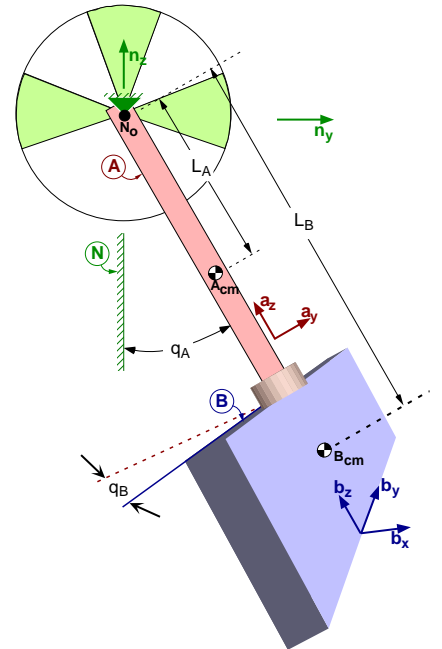
²³In other words, it is *not* possible to write $\cos(x) = a * \sin(x) + b$ when a and b are not functions of x . Another way to see that the coefficients must be zero is to recognize that x may be assigned *any* value. Choosing $x=0$ results in $\cos(x)=1$ and $\sin(x)=0$, while choosing $x=\pi/2$ results in $\cos(x)=0$ and $\sin(x)=1$.

Chapter 3

Introduction to differential equations

3.1 Motivating example: Chaotic plate pendulum

The figure to the right is a schematic representation of a swinging babyboot attached by a shoelace to a rigid support. The mechanical model of the babyboot consists of a thin uniform rod A attached to a fixed support N by a revolute joint, and a uniform plate B connected to A with a second revolute joint so that B can rotate freely about A 's axis. Note: The revolute joints' axes are *perpendicular*, not parallel.



Modeling considerations

- The plate, rod, and support are rigid.
- The revolute joints are frictionless.
- There is no slop or flexibility in the revolute joints.
- The Earth is a Newtonian reference frame.
- Air resistance is negligible.
- The force due to Earth's gravitation is uniform and constant.
- Other distance forces, e.g., electromagnetic and gravitational forces, are negligible.

The following **identifiers** are useful in describing this system.

Quantity	Symbol	Type	Value
Local gravitational constant	g	constant	9.81 m/sec ²
Distance between N_o and A_{cm}	L_A	constant	7.5 cm
Distance between N_o and B_{cm}	L_B	constant	20 cm
Mass of A	m^A	constant	0.01 kg
Mass of B	m^B	constant	0.1 kg
Central moment of inertia of A for \mathbf{a}_x	I_x^A	constant	0.05 kg*cm ²
Central moment of inertia of B for \mathbf{b}_x	I_x^B	constant	2.5 kg*cm ²
Central moment of inertia of B for \mathbf{b}_y	I_y^B	constant	0.5 kg*cm ²
Central moment of inertia of B for \mathbf{b}_z	I_z^B	constant	2.0 kg*cm ²
Angle associated with rotation of A in N	q_A	variable	varies
Angle associated with rotation of B in A	q_B	variable	varies

The *differential equations* governing this system's motion are

$$\ddot{q}_A = \frac{2\dot{q}_A\dot{q}_B\sin(q_B)\cos(q_B)(I_x^B - I_y^B) - (m^A L_A + m^B L_B)g\sin(q_A)}{I^A + m^A L_A^2 + m^B L_B^2 + I_x^B \cos^2(q_B) + I_y^B \sin^2(q_B)}$$

$$\ddot{q}_B = \frac{-\dot{q}_A^2 \sin(q_B)\cos(q_B)(I_x^B - I_y^B)}{I_z^B}$$

The solution to these differential equations is interesting because it reveals that this simple system is capable of strange, non-intuitive motion.¹ For certain initial values of q_A , the motion of plate B is well-behaved and “stable”. Alternately, for other initial values of q_A , the motion of B is “chaotic,” meaning that a small variation in the initial value of q_B or inaccuracies in numerical integration lead to dramatically different results (these differential equations have been used to test the accuracy of numerical integrators).

For example, the “stable” simulation results in Figure 3.1 correspond to an initial value of $q_A(0)=45^\circ$ together with either $q_B(0)=0.5^\circ$ or $q_B(0)=1.0^\circ$. The “chaotic” simulation results in Figure 3.2 show that q_B is *very* sensitive to initial values. A 0.5° change in the value of $q_B(0)$ results in a more than 2000° difference in the value of $q_B(t=10)$!

The following chart and figure shows the regions of stability (white) and instability (grey) that are associated with this system.

Initial value of q_A	Stability
$0^\circ \leq q_A(0) \leq 71.3^\circ$	stable (white)
$71.4^\circ \leq q_A(0) \leq 111.77^\circ$	unstable (grey)
$111.78^\circ \leq q_A(0) \leq 159.9^\circ$	stable (white)
$160.0^\circ \leq q_A(0) \leq 180.0^\circ$	unstable (grey)

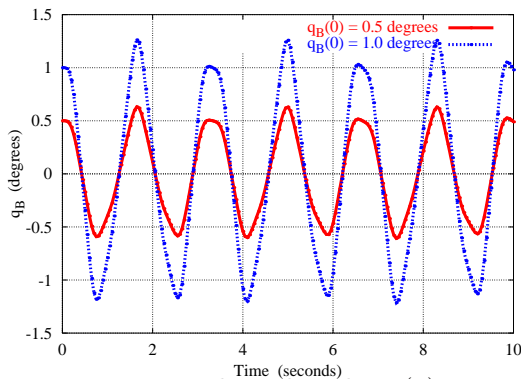
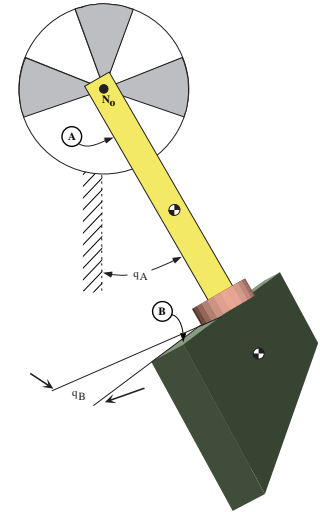


Figure 3.1: Released with $q_A(0)=45^\circ$

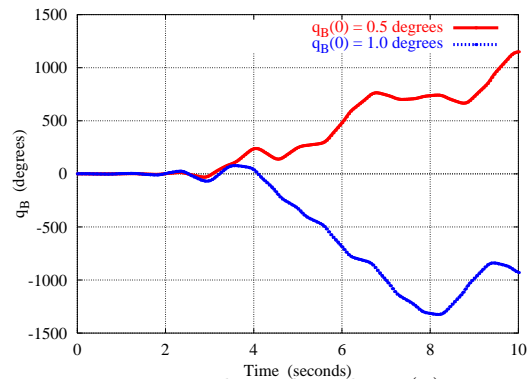


Figure 3.2: Released with $q_A(0)=90^\circ$

3.2 Why we care about differential equations

Differential equations predict a wide range of physical phenomenon, including the motion of particles and bodies, the flow of electricity in circuits, the flow of heat through conduction, convection, and radiation, the flow of fluids at low speeds (laminar flow), as well as the rate of cell growth and cell decay in humans, plants, and animals.

¹More information about this problem can be found in “Mechanical Demonstration of Mathematical Stability and Instability”, International Journal of Engineering Education (also called the Journal of Mechanical Engineering Education), Vol. 2, No. 4, 1974, pp. 45-47, by Thomas R. Kane. The solution to this problem is available by visiting www.autolev.com, selecting **Sample Problems**, and clicking on **Babyboot**.

Studying analytical solutions to differential equations is important for stability analysis, control system design, and insights into certain types of physical phenomenon, e.g., linear vibrations. It also give insights into designing numerical algorithms for solving nonlinear differential equations.

3.3 Why classify differential equations

It is important to classify differential equations because the solution technique depends on the classification. A relevant analogy to classification of differential equations is classifying *algebraic* equations as *linear* or *nonlinear*, because the techniques for solving nonlinear algebraic equations (e.g., the secant method or the Newton-Rhapson method) are different from the solution techniques for solving linear algebraic equations (e.g., Gaussian elimination and Cramer’s rule).

3.4 Independent, dependent, and specified variables

To classify a differential equation, four types of mathematical quantities need to be identified.

- An **independent variable** is a quantity that varies independently, i.e., it is not controlled by (or depend on) other variables. For many dynamic systems there is one independent variable, and it is either t (time) or s (a complex variable used in conjunction with the Laplace transform).
- The symbol x is a **dependent variable** if its value depends on the independent variable and its dependence is considered to be **unknown**, e.g., it is governed by a differential equation.
- The symbol x is a **specified variable** if its dependence on the independent variable can be considered to be **known**.² For example, x is specified if it is a known function of time such as $x=\sin(t)$.
- The symbol c is a **constant** if its value does not change, i.e., its value does not depend on the independent variable.

3.5 Examples of classification of equations

In the examples that follow, assume x and y are dependent variables and t is an independent variable.

1. Differential and algebraic equations

An equation is called a **differential equation** if it contains a derivative of a dependent variable (an unknown) with respect to an independent variable, otherwise it is called a **algebraic equation**. In the examples that follow, circle either *differential* or *algebraic*.

$\frac{dy}{dt} = 7$	differential/algebraic equation
$3 * y = 7$	differential/algebraic equation
$\frac{\partial^2 y}{\partial t^2} + \frac{\partial y}{\partial s} = 0$	differential/algebraic equation
$7y^4 + \cos(t) \sin(y)^3 = \tan(t) + 99$	differential/algebraic equation

- **Ordinary and partial derivatives**

When a dependent variable (e.g., x) is a function of only *one* independent variable (e.g., t), then the derivative of x with respect to t is called the **ordinary derivative** derivative of x with respect to t , and is denoted $\frac{dx}{dt}$ or \dot{x} or x' . Alternately, if x is a function of *two or more* independent variables (e.g., s and t), then the derivative of x with respect to t is called the **partial derivative** of x with respect to t , and is denoted $\frac{\partial x}{\partial t}$.³

When an equation contains a partial derivative of a dependent variable with respect to an independent variable, the equation is called a **partial differential equation** (abbreviated PDE).

²Another name for specified is **prescribed**.

³This textbook only considers systems with one independent variable.

If an equation contains only ordinary derivatives (of dependent variables with respect to an independent variable), the equation is called a **ordinary differential equation** (abbreviated **ODE**). In the examples that follow, circle either *ordinary* or *partial*.

$\frac{dx}{dt} = e^t$	ordinary/partial differential equation
$\frac{\partial y}{\partial t} = 0$	ordinary/partial differential equation
$\frac{d^2x}{dt^2} + \sin(t) \left(\frac{dx}{dt}\right)^2 + \tan(x) = e^t$	ordinary/partial differential equation
$\frac{\partial^2 y}{\partial t^2} + \frac{\partial^2 y}{\partial s^2} = 0$	ordinary/partial differential equation

• **First-order, second-order, third-order, ...**

A differential equation is classified by the order of the highest derivative of a dependent variable that appears in the equation.⁴ In the following examples, fill in the blanks with the proper order, e.g., 1st-order, 2nd-order, etc.

$\sin(t) \left(\frac{dx}{dt}\right)^3 + \tan(x) = 0$	[]-order ODE
$\frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^3 + \tan(x) = 0$	[]-order ODE
$\frac{\partial^3 y}{\partial t^3} + \frac{\partial y}{\partial s} = 0$	[]-order PDE

• **Constant-coefficient, periodic-coefficient, and variable coefficient**

After identifying the factors in the differential equation that contain a dependent variable, one may inspect each of the *coefficients* of the factors to determine if the coefficients are constant, periodic, or variable. A differential equation is called **constant-coefficient** if *all* the coefficients are constants. If *all* the coefficients are constant or periodic, the differential equation is called **periodic-coefficient**.⁵ When one or more of the coefficients is a non-periodic function of the independent variable, the differential equation is called **variable-coefficient**.

In the examples that follow, fill in the blanks with *constant-coefficient*, *periodic-coefficient*, or *variable-coefficient*.

$\ddot{x} + 3 * x - \sin(t) = 0$	[]-coefficient, 2 nd -order, ODE
$\ddot{x} + t * x = 0$	[]-coefficient, 2 nd -order, ODE
$3\ddot{x} + x * \dot{x} + 7 * \sin(x) * x = 0$	[]-coefficient, 2 nd -order, ODE
$\sin(t) * \dot{x} + x = 0$	[]-coefficient, 1 st -order, ODE

2. **Homogeneous and inhomogeneous**

To determine if an algebraic or differential equation is **homogeneous** or **inhomogeneous**, the equation is written so that all the dependent variables and their derivatives (e.g., x , \dot{x} , \ddot{x} , y , \dot{y} , \ddot{y}) are on the left-hand side of the equation and all the remaining terms (e.g., those terms not containing x , \dot{x} , \ddot{x} , y , \dot{y} , \ddot{y}) are on the right-hand side of the equation. The term on the right-hand side of the equation will either be 0, a constant, or a function of the independent variables. If the right-hand side is 0, the equation is called *homogeneous*, otherwise it is called *inhomogeneous*.

In the examples that follow, circle either *homogeneous* or *inhomogeneous*.

⁴A physical system that is governed by a n^{th} -order differential equation is called an n^{th} -order system.

⁵Periodic-coefficient differential equations are a special subset of variable-coefficient differential equations whose solution technique involves Floquet theory.

$x + y = 0$	homogeneous/inhomogeneous algebraic equation
$x + y = t$	homogeneous/inhomogeneous algebraic equation
$t^2 \ddot{x} + \sin(t) \dot{x}^2 + x = 0$	homogeneous/inhomogeneous ODE
$\ddot{x} + 3 = 0$	homogeneous/inhomogeneous ODE

3. Linear and nonlinear

To determine if an expression is *linear* or *nonlinear*, one must first ask the question “*Linear in what?*”. In the context of differential equations, a differential equation is *linear* when it is linear in the dependent variables and their derivatives. e.g., linear in $x, \dot{x}, \ddot{x}, y, \dot{y}, \ddot{y}$.

For example, a second-order ODE with one dependent variable x is said to be a *linear differential equation* when it can be expressed as

$$c_0 + c_1 * x + c_2 * \dot{x} + c_3 * \ddot{x} = 0$$

where c_i ($i=0,1,2,3$) are independent of x, \dot{x} , and \ddot{x} . If the differential equation cannot be expressed this way, it is a *nonlinear differential equation*. A first-order ODE that has three dependent variables x, y , and z is *linear* when it can be expressed as

$$c_0 + c_1 * x + c_2 * \dot{x} + c_3 * y + c_4 * \dot{y} + c_5 * z + c_6 * \dot{z} = 0$$

where c_i ($i=0,1,2,3,4,5,6$) are independent of $x, \dot{x}, y, \dot{y}, z$, and \dot{z} . If the differential equation cannot be expressed this way, it is a *nonlinear* differential equation.⁶

In the following examples, circle either *linear* or *nonlinear*.

$[t^2 + \sin(t)]x + \tan(t) = 4$	linear/nonlinear inhomogeneous algebraic equation
$x^2 + \sin(x) = 4$	linear/nonlinear inhomogeneous algebraic equation
$t^2 \ddot{x} + \sin(t) \dot{x} + x = 0$	linear/nonlinear homogeneous ODE
$\ddot{x} + \sin(x) = 0$	linear/nonlinear homogeneous ODE

4. Uncoupled and coupled

To determine if a set of equations is *coupled* or *uncoupled*, one must first ask “*Coupled in what?*”. In the context of algebraic equations, a *set* of equations is said to be coupled when the same dependent variable appears in more than one equation. A set of two or more differential equations is said to be coupled when a dependent variable or one of its derivatives (e.g., x, \dot{x}, \ddot{x}), appears in more than one equation.

In the examples that follow, fill in the blanks with *coupled* or *uncoupled*.

$t^2 \cos(x^3) + \sin(t)x^2 + x = 0$	<input type="text"/>	nonlinear inhomogeneous algebraic
$y + t^3 \sin(y) - 9 = 0$		
$x + y = 0$	<input type="text"/>	linear inhomogeneous algebraic
$x = 7$		
$t^2 \ddot{x} + \sin(t) \dot{x}^2 + x = 0$	<input type="text"/>	nonlinear inhomogeneous differential
$\ddot{y} + t^3 \sin(y) - \tan(t) = 0$		

⁶Solving *nonlinear* equations (whether differential or algebraic) is usually significantly *more difficult* than solving linear equations. Even determining the *number* of solutions to nonlinear equations can be difficult or impossible, e.g., a nonlinear algebraic equation may have no solution, one solution, two solutions, seventeen solutions, or infinite solutions. For example, try to find the values of x that satisfy $x^2 + \sin(x) = 4$. The solutions are $x=1.7359755$ and $x=-2.1936735$ and can be obtained by a variety of methods, e.g., Newton-Rhapson or graphing.

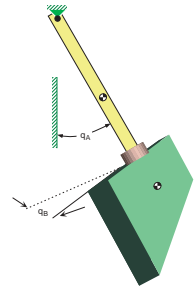
$$\begin{aligned}\dot{x} + \dot{y} &= 0 \\ \ddot{x} &= 0\end{aligned}$$

 linear homogeneous differential

3.6 Example of classification of differential equations

The equations governing the motion of the system described in Section 3.1 are

$$\begin{aligned}\ddot{q}_A &= \frac{2 [508.89 \sin(q_A) - \sin(q_B) \cos(q_B) \dot{q}_A \dot{q}_B]}{-21.556 + \sin(q_B)^2} \\ \ddot{q}_B &= -\sin(q_B) \cos(q_B) \dot{q}_A^2\end{aligned}$$



This set of equations is classified as (circle the appropriate qualifiers)

Uncoupled	Linear	Homogeneous	Constant-coefficient	1st-order	Algebraic
Coupled	Nonlinear	Inhomogeneous	Variable-coefficient	2nd-order	Differential

It is worth noting that in general, an algebraic equation is *good* if it is linear. A differential equation is *good* if it is linear, homogeneous, constant-coefficient, first-order, and ordinary.⁷

3.7 What is a vibration?

The word **vibration** is usually associated with some manner of motion, but the word has different meanings in various contexts. One precise definition is **“vibrations are motions governed by a certain class of differential equations”**. This definition is broad enough to encapsulate vibrations governed by differential equations with periodic time-dependent coefficients and vibrations governed by partial differential equations.⁸ A special class of vibrations are vibrations whose motions are governed by a linear, constant-coefficient ODE of the form

$$M \ddot{X} + B \dot{X} + K X = F(t)$$

where X is a $n \times 1$ (n is a positive integer) matrix of dependent variables x_1, x_2, \dots, x_n ; M , B , and K are $n \times n$ matrices of constants; and $F(t)$ is a $n \times 1$ matrix whose elements may be constant or a function of time. For example,

$$X \triangleq \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad M \triangleq \begin{bmatrix} m_{11} & m_{12} & \dots & m_{1n} \\ m_{21} & m_{22} & \dots & m_{2n} \\ \cdot & \cdot & \dots & \cdot \\ m_{n1} & m_{n2} & \dots & m_{nn} \end{bmatrix} \quad F(t) \triangleq \begin{bmatrix} f_1(t) \\ f_2(t) \\ \cdot \\ f_n(t) \end{bmatrix}$$

3.8 Solution techniques for ordinary differential equations

There are a variety of techniques for solving ODEs, and several are listed below together with related sections and/or chapters of the textbook.

⁷A **good** equation means that it is easy to manipulate and obtain a solution. A **bad** equation means that instead of enjoying a basketball game with your friends, you are struggling with difficult mathematics.

⁸With this broad definition, solving a linear, constant-coefficient, ODE is identical to doing a vibrations problem. For many practical purposes, having studied differential equations in **dynamic systems** is equivalent to having studied **vibrations**.

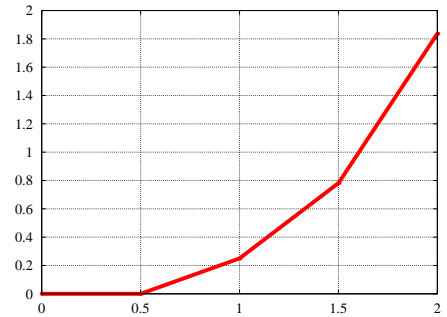
Technique	Section or chapter
Numerical solution	Section 2.13
Separation of variables	Chapter 4
Assumed solutions of the form e^{pt}	Chapters 6, 7, 26, 27
Laplace transforms	Chapter 19
Small approximations	Chapter 21
Taylor series approximations	Chapter 22
Trial and error	

3.9 Numerical solution of ordinary differential equations - concepts

Consider the nonlinear, inhomogeneous, 1st-order, ODE $\frac{dy}{dt} = y^2 + t$.

There are many numerical integration algorithms for solving this ODE (e.g., Euler integration, predictor-corrector, and Runge-Kutta).⁹ Robust numerical methods for solving ODEs with **MotionGenesis** and Matlab are described in Section 2.13. The following table shows how to use **Euler integration** (one of the simplest methods) to solve for the first 2.0 seconds of $y(t)$ when the initial value of y is $y(t=0) = 0$.

t	$y(t)$	$\dot{y}(t) = y(t)^2 + t$	$y(t+\Delta t) \approx y(t) + \dot{y}(t)*\Delta t$
0.0	0	0	0
0.5	0	0.5	0.25
1.0	0.25	1.0625	0.78125
1.5	0.78125	2.110352	1.836426
2.0	1.836426	5.37246	4.522656



⁹Homework 5.5 requires a numerical solution to a differential equation.