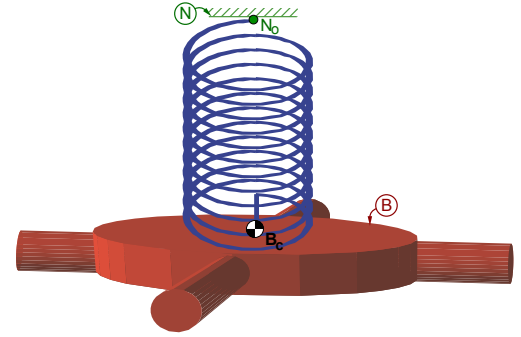




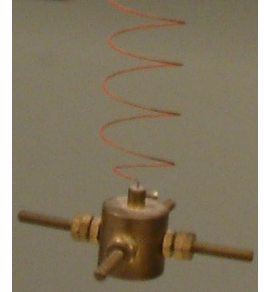
## F.2 (31 pts.) Coupled motions of Wilbur-force pendulum

Shown to the right is a rigid body  $B$  that is attached to a spring at  $B_c$ , the mass center of  $B$ . The other end of the spring is attached to point  $N_o$  which is fixed in a Newtonian reference frame  $N$ . The equations governing this model of the system are

$$\begin{aligned} m \ddot{x} + k_x x + k_c \theta &= 0 \\ I \ddot{\theta} + k_c x + k_\theta \theta &= 0 \end{aligned}$$



Quantity	Symbol	Type
Mass of $B$	$m$	constant
Central moment of inertia of $B$ about vertical	$I$	constant
Linear spring constant modeling extensional flexibility	$k_x$	constant
Linear spring constant modeling torsional flexibility	$k_\theta$	constant
Linear spring constant modeling coupled flexibility	$k_c$	constant
Translational stretch of spring from equilibrium	$x$	dependent variable
Rotational stretch of spring from equilibrium	$\theta$	dependent variable



- (a) (2 pts.) Write the ODEs in the matrix form  $M \ddot{X} + B \dot{X} + K X = 0$ , where  $X \triangleq \begin{bmatrix} x \\ \theta \end{bmatrix}$ .

**Result:**

$$\begin{bmatrix} m & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} k_x & k_c \\ k_c & k_\theta \end{bmatrix} \begin{bmatrix} x \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- (b) (9 pts.) The solution to the previous set of ODEs has the form  $X(t) = U * e^{pt}$  where  $p$  is a constant (to-be-determined) and  $U$  is a **non-zero**  $2 \times 1$  matrix of constants (to-be-determined). Fill in the two blanks in the following polynomial equation that governs the values of  $\lambda \triangleq -p^2$ .  
Note: the blanks only involve  $m, I, k_x, k_\theta, k_c$ .

**Result:**

$$\lambda^2 + \left( \frac{-k_x}{m} + \frac{-k_\theta}{I} \right) * \lambda + \frac{k_x k_\theta - k_c^2}{m I} = 0$$

- (c) (6 pts.) For certain values of  $m$ ,  $I$ ,  $k_x$ ,  $k_\theta$ , and  $k_c$ , the matrix  $A \triangleq M^{-1}K = \begin{bmatrix} 10 & 1 \\ 1 & 10 \end{bmatrix}$ . Find the eigenvalues and corresponding eigenvectors of  $A$ .

**Result:**

$$\lambda_1 = 9 \quad U_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

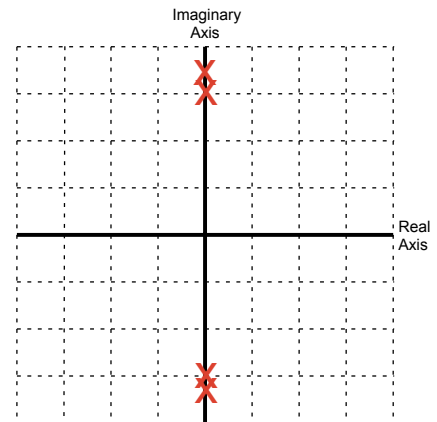
$$\lambda_2 = 11 \quad U_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(2 pts.) Calculate the values of  $p_1, p_2, p_3, p_4$  and draw their locations in the complex plane.

(d)

$$p_{1,2} = \pm \sqrt{-\lambda_1} = \pm \sqrt{-9} = \pm 3i$$

$$p_{3,4} = \pm \sqrt{-\lambda_2} = \pm \sqrt{-11} = \pm 3.32i$$



- (e) (1 pt.) The solution is **stable**/**neutrally stable**/**unstable**.
- (f) (4 pts.) Write the solution for  $X(t)$  in terms of the yet-to-be-determined constants  $c_1, c_2, c_3, c_4$ , the sine and cosine functions, and  $t$ .

**Result:**

$$\begin{bmatrix} x(t) \\ \theta(t) \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \{c_1 \sin(3t) + c_2 \cos(3t)\} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \{c_3 \sin(3.32t) + c_4 \cos(3.32t)\}$$

- (g) (2 pts.) Determine  $c_1, c_2, c_3, c_4$  when  $x(0) = 0.2, \theta(0) = 0, \dot{x}(0) = 0,$  and  $\dot{\theta}(0) = 0.$

**Result:**

$$c_1 = 0 \quad c_2 = -0.1 \quad c_3 = 0 \quad c_4 = 0.1$$

- (h) (2 pts.) Using the aforementioned initial values, write explicit solutions for  $x(t)$  and  $\theta(t).$

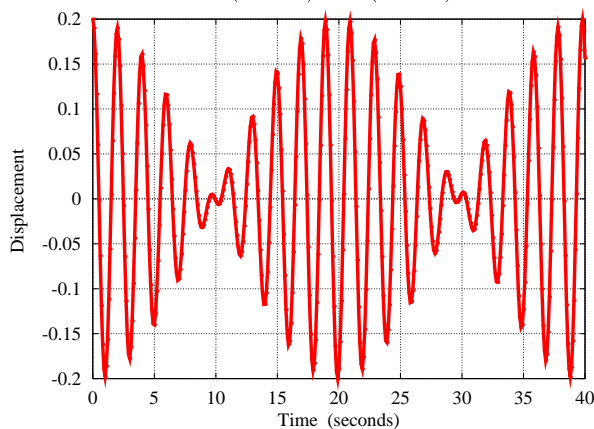
**Result:**

$$x(t) = 0.1 \cos(3t) + 0.1 \cos(3.32t)$$

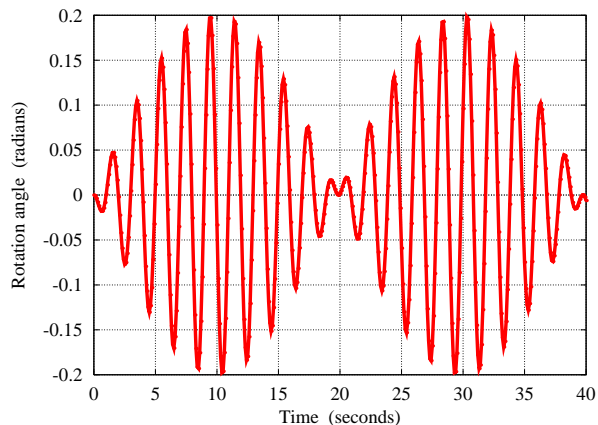
$$\theta(t) = -0.1 \cos(3t) + 0.1 \cos(3.32t)$$

- (i) (1 pt.) Using trigonometric identities, equation (2.19), and  $\cos(a) = -\cos(a+\pi),$  show that your previous solution for  $x(t)$  and  $\theta(t)$  is

$$\begin{aligned} x(t) &= 0.2 \sin(-0.16t + \frac{\pi}{2}) \sin(3.16t + \frac{\pi}{2}) \\ &= 0.2 \cos(0.16t) \cos(3.16t) \end{aligned}$$



$$\begin{aligned} \theta(t) &= 0.2 \sin(-0.16t) \sin(3.16t) \\ &= 0.2 \sin(0.16t + \pi) \sin(3.16t) \end{aligned}$$



- (j) (2 pts.) Give an interpretation of the time-behavior of  $x(t)$  and  $\theta(t).$

**Result:**

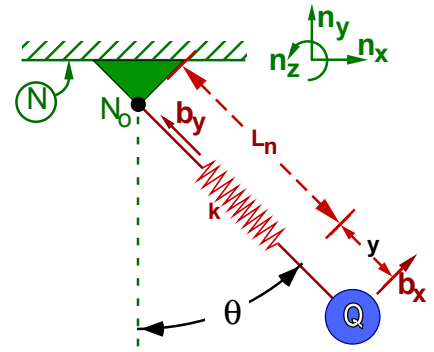
Both  $x(t)$  and  $\theta(t)$  exhibit the *beat phenomena* with a high-frequency of  $3.16 \frac{\text{rad}}{\text{sec}}$  and a low-frequency of  $0.16 \frac{\text{rad}}{\text{sec}}.$

Since  $x(t)$  and  $\theta(t)$  are *coupled*,  $x$ 's maximum amplitude coincides with  $\theta$ 's minimum amplitude (and vice-versa).

**F.3 (38 pts.) Linearization of equations of motion for a swinging spring**

A straight, massless, linear spring connects a particle  $Q$  to a point  $N_o$  which is fixed in a Newtonian reference frame  $N$ . The system identifiers and governing equations are shown below.

Quantity	Identifier	Type
Local gravitational constant	$g$	constant
Mass of $Q$	$m$	constant
Natural spring length	$L_n$	constant
Linear spring constant	$k$	constant
Spring stretch	$y$	variable
"Pendulum angle"	$\theta$	variable



$$m(L_n + y)\ddot{\theta} + 2m\dot{\theta}\dot{y} + mg\sin(\theta) = 0$$

$$m\ddot{y} - m(L_n + y)\dot{\theta}^2 + ky - mg\cos(\theta) = 0$$

- (a) (1 pt.) Classify the previous equations by picking the relevant qualifiers from the following list.

Uncoupled	Linear	Homogeneous	Constant-coefficient	1st-order	Algebraic
Coupled	Nonlinear	Inhomogeneous	Variable-coefficient	2nd-order	Differential

- (b) (3 pts.) Find the condition on  $y_{\text{nom}}$  such that  $y = y_{\text{nom}}$  (a constant) and  $\theta = \theta_{\text{nom}} = 0$  satisfy the nonlinear ODEs.

Result:

$$y_{\text{nom}} = \frac{mg}{k}$$

- (c) (10 pts.) Using a Taylor series, linearize the differential equations in perturbations about this nominal solution. In other words, define perturbations of  $y$  and  $\theta$  as

$$\tilde{y} \triangleq y - y_{\text{nom}} \quad (y_{\text{nom}} \text{ is constant}) \qquad \tilde{\theta} \triangleq \theta - \theta_{\text{nom}} \quad (\theta_{\text{nom}} = 0)$$

and form a set of differential equations which are linear in  $\tilde{y}, \dot{\tilde{y}}, \ddot{\tilde{y}}, \tilde{\theta}, \dot{\tilde{\theta}}, \ddot{\tilde{\theta}}$ .

Result:

$$m(L_n + y_{\text{nom}})\ddot{\tilde{\theta}} + mg\tilde{\theta} = 0$$

$$m\ddot{\tilde{y}} + k\tilde{y} = 0$$

- (d) (7 pts.) Replace  $y$  and  $\theta$  in the original nonlinear ODEs with

$$y = \tilde{y} + y_{\text{nom}} \quad (y_{\text{nom}} \text{ is the constant determined earlier}) \quad \theta = \tilde{\theta} + \theta_{\text{nom}} \quad (\theta_{\text{nom}} = 0)$$

Assuming  $\tilde{y}$ ,  $\dot{\tilde{y}}$ ,  $\ddot{\tilde{y}}$  and  $\tilde{\theta}$ ,  $\dot{\tilde{\theta}}$ , and  $\ddot{\tilde{\theta}}$  are small and using the **small-approximations technique** of Chapter 20, form a set of ODEs that are linear in  $\tilde{y}$ ,  $\dot{\tilde{y}}$ ,  $\ddot{\tilde{y}}$ ,  $\tilde{\theta}$ ,  $\dot{\tilde{\theta}}$ ,  $\ddot{\tilde{\theta}}$ .

**Result:**

$$\begin{aligned} m(L_n + y_{\text{nom}})\ddot{\tilde{\theta}} + mg\tilde{\theta} &= 0 \\ m\ddot{\tilde{y}} + k\tilde{y} &= 0 \end{aligned}$$

- (e) (1 pt.) Classify your equations in part (3d) by picking the relevant qualifiers from the following list.

<b>Uncoupled</b>	<b>Linear</b>	<b>Homogeneous</b>	<b>Constant-coefficient</b>	1st-order	Algebraic
Coupled	Nonlinear	Inhomogeneous	Variable-coefficient	<b>2nd-order</b>	<b>Differential</b>

- (f) (1 pt.) The linearized ODEs resulting from the Taylor series technique are identical to those obtained from the small-approximations technique. **True/False.**
- (g) (5 pts.) Find analytical solutions for  $\tilde{y}(t)$  and  $\tilde{\theta}(t)$  when  $g = 9.8 \text{ m/sec}^2$ ,  $m = 1 \text{ kg}$ ,  $L_n = 0.5 \text{ m}$ ,  $k = 100 \text{ n/m}$ ,  $\tilde{y}(0) = 0.1 \text{ m}$ ,  $\tilde{\theta}(0) = 1^\circ$ , and  $\dot{\tilde{y}}(0) = \dot{\tilde{\theta}}(0) = 0$

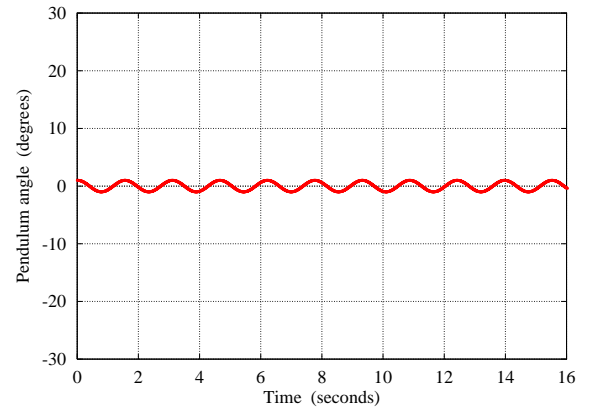
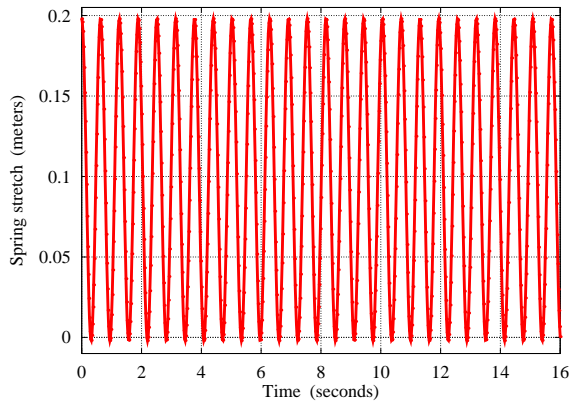
$$\tilde{y}(t) = 0.1 \cos(\sqrt{\frac{k}{m}} t) = 0.1 \cos(10 t)$$

$$\tilde{\theta}(t) = 1^\circ \cos(\sqrt{\frac{g}{L_n + y_{\text{nom}}}} t) = 1^\circ \cos(\sqrt{\frac{gk}{L_n k + mg}} t) = 1^\circ \cos(4.05 t)$$

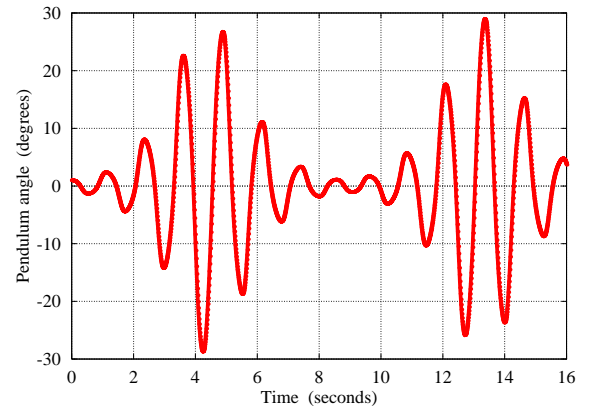
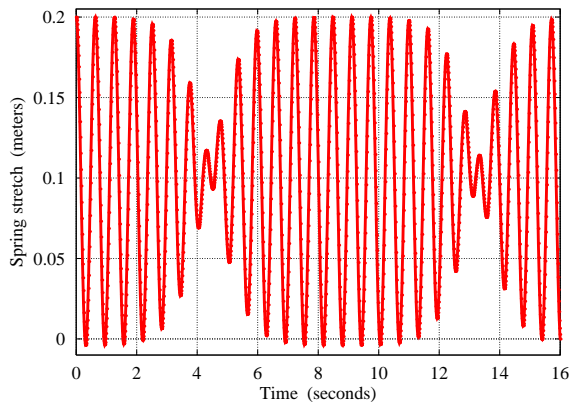
- (h) (4 pts.) Make a **rough** sketch of your linearized solutions for  $y(t)$  and  $\theta(t)$  for  $0 \leq t \leq 16$  sec, showing the relevant characteristics, e.g., amplitude, frequency, growth, decay, etc.

$$y(t) \approx \tilde{y}(t) + y_{\text{nom}} = 0.1 \cos(10t) + 0.098$$

$$\theta(t) \approx \tilde{\theta}(t) + \theta_{\text{nom}} = 1^\circ \cos(4.05t)$$



- (i) (6 pts.) The computer solution for  $y(t)$  and  $\theta(t)$  to the original nonlinear ODEs for  $0 \leq t \leq 16$  sec are plotted below. Comment on the similarities and differences between the plots you produced with the ones below. Give a reason for the similarities and differences.



Similarities: Frequency, maximum amplitude

Similarities: Frequency

Differences: Beat phenomenon, varying amplitude

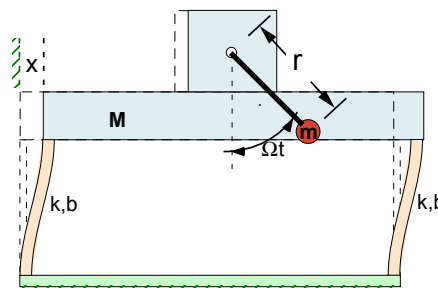
Differences: Amplitude, beat phenomenon

Reason for differences: Coupled and nonlinear

Reason for differences: Coupled and nonlinear

**F.4 (29 pts.) Dynamic response for an air-conditioner on a building**

An air conditioner is bolted to the roof of a one story building. The air conditioner's motor is unbalanced and its eccentricity is modeled as a particle of mass  $m$  attached to the distal end of a rigid rod of length  $r$ . When the motor spins with angular speed  $\Omega$ , it causes the building's roof of mass  $M$  to vibrate. The stiffness and material damping in *each* column that supports the roof is modeled as a linear horizontal spring ( $k$ ) and linear horizontal damper ( $b$ ). Homework 4.5 showed that the ODE governing the horizontal displacement  $x$  of the building's roof is



$$(M+m)\ddot{x} + 2b\dot{x} + 2kx = mr\Omega^2 \sin(\Omega t)$$

- (a) **(3 pts.)** Express  $\omega_n$ ,  $\zeta$ ,  $A$  in terms of  $M$ ,  $m$ ,  $b$ ,  $k$ ,  $r$  so the ODE is

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2x = A\Omega^2 \sin(\Omega t)$$

**Result:**

$$\omega_n = \sqrt{\frac{2k}{M+m}} \quad \zeta = \frac{b}{\sqrt{(M+m)k}} \quad A = \frac{mr}{M+m}$$

- (b) **(2 pts.)** For certain values of  $M$ ,  $m$ ,  $b$ ,  $k$ , and  $r$ , this ODE simplifies to

$$\ddot{x} + 3\dot{x} + 900x = 1 \times 10^{-4} \Omega^2 \sin(\Omega t)$$

Calculate numerical values for  $\omega_n$  and  $\zeta$ .

**Result:**

$$\omega_n = 30 \frac{\text{rad}}{\text{sec}} \quad \zeta = 0.05$$

- (c) **(7 pts.)** Fill in numerical values in the following expressions for  $x_{\text{ss}}(t)$ , the steady-state part of  $x(t)$ .

$\Omega$ ( $\frac{\text{rad}}{\text{sec}}$ )	$x_{\text{ss}}(t)$
20	$7.94 \times 10^{-5} * \sin(20t + -6.843^\circ)$
30	$0.001 * \sin(30t + -90^\circ)$
40	$2.25 \times 10^{-4} * \sin(40t + -170.3^\circ)$

- (d) **(14 pts.)** The building's occupants complain that the roof shakes too much. Comment on the effect small variations of  $M$ ,  $m$ ,  $b$ ,  $k$ , and  $r$  have on:  $\zeta$ ,  $\frac{\Omega}{\omega_n}$ , and the magnitude of  $x_{ss}(t)$ . Fill in each element in the table by writing  $-$  (decreases),  $\mathbf{0}$  (no effect),  $+$  (increases), or  $?$  (if it may decrease *or* increase). For the 2<sup>nd</sup>-to-last column, assume the air conditioner's normal operating speed is  $\Omega = 28 \frac{\text{rad}}{\text{sec}}$ , and for last column, assume  $\Omega = 32 \frac{\text{rad}}{\text{sec}}$ .

	$\zeta$	$\frac{\Omega}{\omega_n}$	$\Omega \approx 28 \frac{\text{rad}}{\text{sec}}$ $ x_{ss}(t) $	$\Omega \approx 32 \frac{\text{rad}}{\text{sec}}$ $ x_{ss}(t) $
Balancing the motor ( $r \rightarrow 0$ )	$\mathbf{0}$	$\mathbf{0}$	$-$	$-$
Increasing the motor speed $\Omega$	$\mathbf{0}$	$+$	$+$	$-$
Decreasing the motor speed $\Omega$	$\mathbf{0}$	$-$	$-$	$+$
Adding mass to the roof (increasing $M$ )	$-$	$+$	$+?$	$?$
Removing mass from the roof	$+$	$-$	$-?$	$?$
Stiffening the support columns (increasing $k$ )	$-$	$-$	$-$	$+$
Adding damping to the columns (increasing $b$ )	$+$	$\mathbf{0}$	$-$	$-$

- (e) **(3 pts.)** List three ways to change the **motor** and minimize the roof shaking.  
 Balance the motor, damp the motor, or operate  $\Omega \gg \omega_n$  or  $\Omega \ll \omega_n$ .