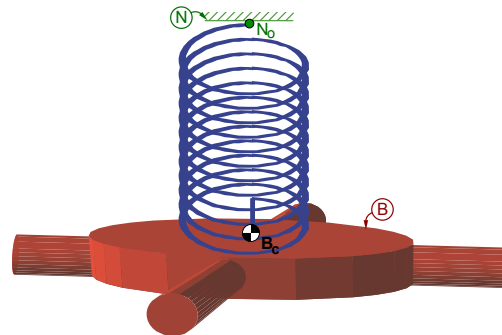


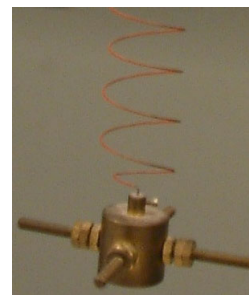
F.2 (31 pts.) Coupled motions of Wilbur-force pendulum

Shown to the right is a rigid body B that is attached to a spring at B_c , the mass center of B . The other end of the spring is attached to point N_o which is fixed in a Newtonian reference frame N . The equations governing this model of the system are

$$\begin{aligned} m \ddot{x} + k_x x + k_c \theta &= 0 \\ I \ddot{\theta} + k_c x + k_\theta \theta &= 0 \end{aligned}$$



Quantity	Symbol	Type
Mass of B	m	constant
Central moment of inertia of B about vertical	I	constant
Linear spring constant modeling extensional flexibility	k_x	constant
Linear spring constant modeling torsional flexibility	k_θ	constant
Linear spring constant modeling coupled flexibility	k_c	constant
Translational stretch of spring from equilibrium	x	dependent variable
Rotational stretch of spring from equilibrium	θ	dependent variable



- (a) (2 pts.) Write the ODEs in the matrix form $M \ddot{X} + B \dot{X} + K X = 0$, where $X \triangleq \begin{bmatrix} x \\ \theta \end{bmatrix}$.

Result:

$$\begin{bmatrix} & \\ & \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} & \\ & \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} & \\ & \end{bmatrix} \begin{bmatrix} x \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- (b) (9 pts.) The solution to the previous set of ODEs has the form $X(t) = U * e^{pt}$ where p is a constant (to-be-determined) and U is a **non-zero** 2×1 matrix of constants (to-be-determined). Fill in the two blanks in the following polynomial equation that governs the values of $\lambda \triangleq -p^2$.
Note: the blanks only involve m, I, k_x, k_θ, k_c .

Result:

$$\lambda^2 + \frac{\quad}{\quad} * \lambda + \frac{\quad}{\quad} = 0$$

- (c) **(6 pts.)** For certain values of m , I , k_x , k_θ , and k_c , the matrix $A \triangleq M^{-1}K = \begin{bmatrix} 10 & 1 \\ 1 & 10 \end{bmatrix}$.

Find the eigenvalues and corresponding eigenvectors of A .

Result:

$$\lambda_1 = \quad U_1 = \begin{bmatrix} \quad \\ \quad 1 \end{bmatrix}$$

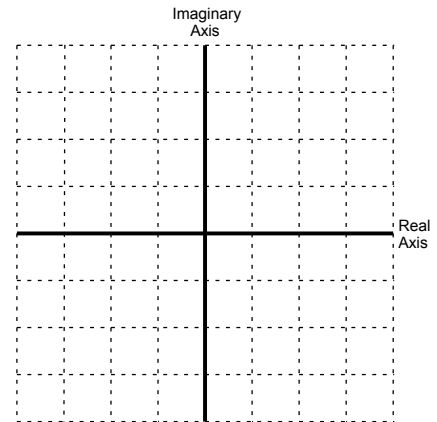
$$\lambda_2 = \quad U_2 = \begin{bmatrix} \quad \\ \quad 1 \end{bmatrix}$$

(2 pts.) Calculate the values of p_1, p_2, p_3, p_4 and draw their locations in the complex plane.

(d)

$$p_{1,2} =$$

$$p_{3,4} =$$



(e) **(1 pt.)** The solution is **stable/neutrally stable/unstable**.

(f) **(4 pts.)** Write the solution for $X(t)$ in terms of the yet-to-be-determined constants c_1, c_2, c_3, c_4 , the sine and cosine functions, and t .

Result:

$$\begin{bmatrix} x(t) \\ \theta(t) \end{bmatrix} = \begin{bmatrix} \quad \\ \quad 1 \end{bmatrix} \left\{ c_1 \sin(\quad t) + c_2 \cos(\quad t) \right\} + \begin{bmatrix} \quad \\ \quad 1 \end{bmatrix} \left\{ c_3 \sin(\quad t) + c_4 \cos(\quad t) \right\}$$

- (g) (2 pts.) Determine c_1, c_2, c_3, c_4 when $x(0) = 0.2$, $\theta(0) = 0$, $\dot{x}(0) = 0$, and $\dot{\theta}(0) = 0$.

Result:

$$c_1 = \quad c_2 = \quad c_3 = \quad c_4 =$$

- (h) (2 pts.) Using the aforementioned initial values, write explicit solutions for $x(t)$ and $\theta(t)$.

Result:

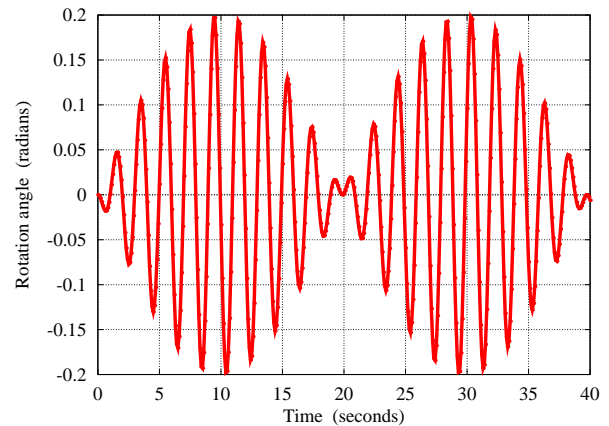
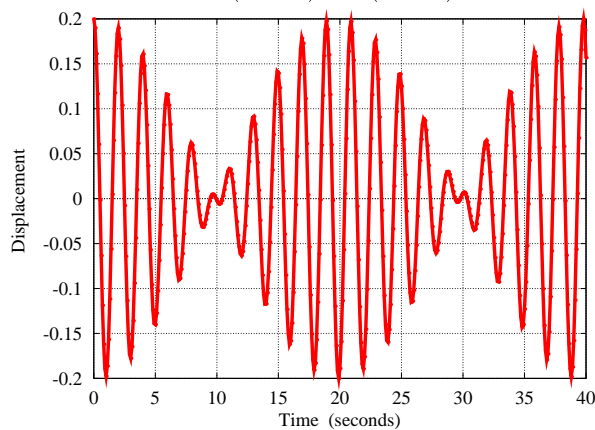
$$x(t) =$$

$$\theta(t) =$$

- (i) (1 pt.) Using trigonometric identities, equation (2.19), and $\cos(a) = -\cos(a + \pi)$, show

$$\begin{aligned} x(t) &= 0.2 \sin(-0.16 t + \frac{\pi}{2}) \sin(3.16 t + \frac{\pi}{2}) \\ &= 0.2 \cos(0.16 t) \cos(3.16 t) \end{aligned}$$

$$\begin{aligned} \theta(t) &= 0.2 \sin(-0.16 t) \sin(3.16 t) \\ &= 0.2 \sin(0.16 t + \pi) \sin(3.16 t) \end{aligned}$$



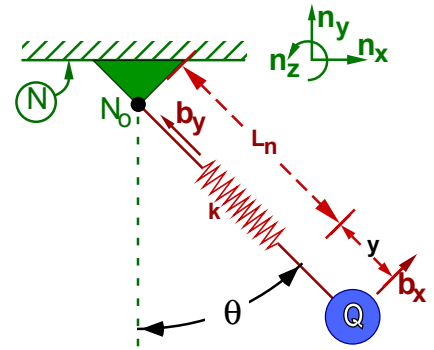
- (j) (2 pts.) Give an interpretation of the time-behavior of $x(t)$ and $\theta(t)$.

Result:

F.3 (38 pts.) Linearization of equations of motion for a swinging spring

A straight, massless, linear spring connects a particle Q to a point N_o which is fixed in a Newtonian reference frame N . The system identifiers and governing equations are shown below.

Quantity	Identifier	Type
Local gravitational constant	g	constant
Mass of Q	m	constant
Natural spring length	L_n	constant
Linear spring constant	k	constant
Spring stretch	y	variable
“Pendulum angle”	θ	variable



$$m(L_n + y)\ddot{\theta} + 2m\dot{\theta}\dot{y} + mg\sin(\theta) = 0$$

$$m\ddot{y} - m(L_n + y)\dot{\theta}^2 + ky - mg\cos(\theta) = 0$$

- (a) (1 pt.) Classify the previous equations by picking the relevant qualifiers from the following list.

Uncoupled	Linear	Homogeneous	Constant-coefficient	1st-order	Algebraic
Coupled	Nonlinear	Inhomogeneous	Variable-coefficient	2nd-order	Differential

- (b) (3 pts.) Find the condition on y_{nom} such that $y = y_{nom}$ (a constant) and $\theta = \theta_{nom} = 0$ satisfy the nonlinear ODEs.

Result:

$$y_{nom} =$$

- (c) (10 pts.) Using a Taylor series, linearize the differential equations in perturbations about this nominal solution. In other words, define perturbations of y and θ as

$$\tilde{y} \triangleq y - y_{nom} \quad (y_{nom} \text{ is constant}) \quad \tilde{\theta} \triangleq \theta - \theta_{nom} \quad (\theta_{nom} = 0)$$

and form a set of differential equations which are linear in $\tilde{y}, \dot{\tilde{y}}, \ddot{\tilde{y}}, \tilde{\theta}, \dot{\tilde{\theta}}, \ddot{\tilde{\theta}}$.

Result:

$$= 0$$

$$= 0$$

- (d) **(7 pts.)** Replace y and θ in the original nonlinear ODEs with

$$y = \tilde{y} + y_{\text{nom}} \quad (y_{\text{nom}} \text{ is the } \mathbf{constant} \text{ determined earlier}) \qquad \theta = \tilde{\theta} + \theta_{\text{nom}} \quad (\theta_{\text{nom}} = 0)$$

Assuming \tilde{y} and $\tilde{\theta}$ are small and using the **small-approximations technique** of Chapter 20, form a set of ODEs that are linear in $\tilde{y}, \dot{\tilde{y}}, \ddot{\tilde{y}}, \tilde{\theta}, \dot{\tilde{\theta}}, \ddot{\tilde{\theta}}$.

Result:

$$= 0$$

$$= 0$$

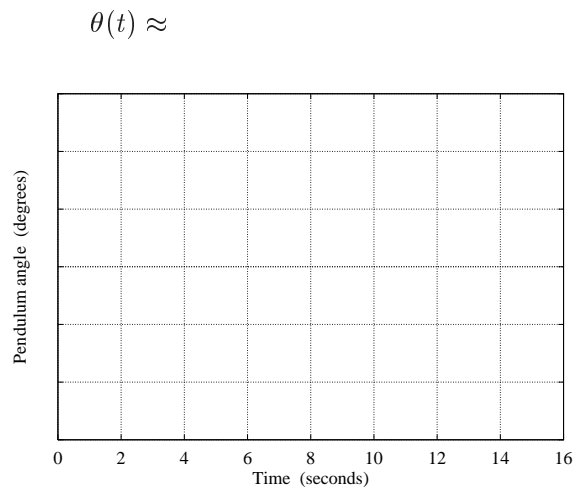
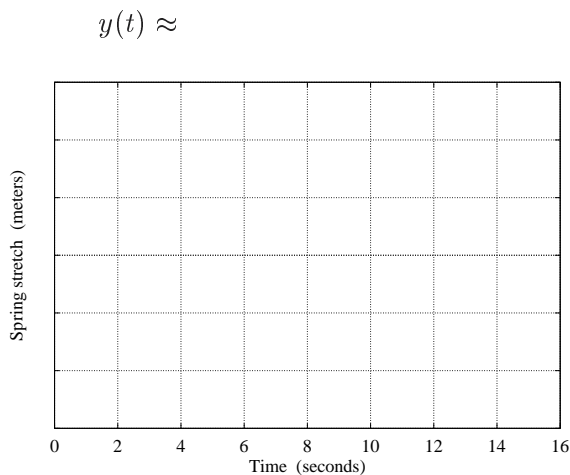
- (e) **(1 pt.)** Classify your equations in part (3d) by picking the relevant qualifiers from the following list.

Uncoupled	Linear	Homogeneous	Constant-coefficient	1st-order	Algebraic
Coupled	Nonlinear	Inhomogeneous	Variable-coefficient	2nd-order	Differential

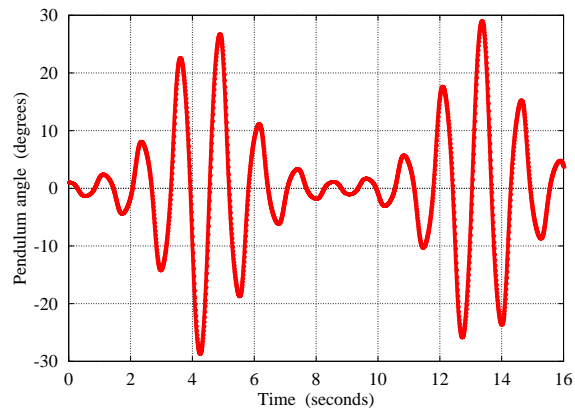
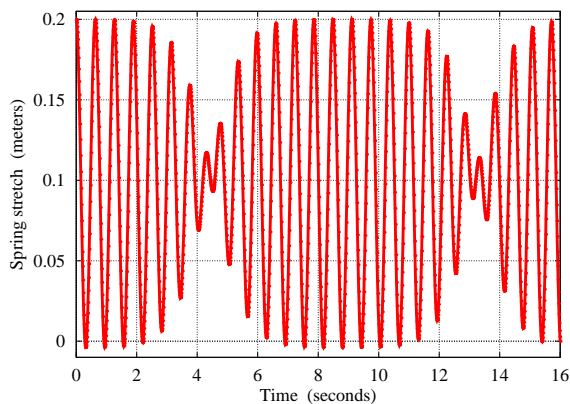
- (f) **(1 pt.)** The linearized ODEs resulting from the Taylor series technique are identical to those obtained from the small-approximations technique. **True/False.**
- (g) **(5 pts.)** Find analytical solutions for $\tilde{y}(t)$ and $\tilde{\theta}(t)$ when $g = 9.8 \text{ m/sec}^2$, $m = 1 \text{ kg}$, $L_n = 0.5 \text{ m}$, $k = 100 \text{ n/m}$, $\tilde{y}(0) = 0.1 \text{ m}$, $\tilde{\theta}(0) = 1^\circ$, and $\dot{\tilde{y}}(0) = \dot{\tilde{\theta}}(0) = 0$

$$\begin{aligned} \tilde{y}(t) &= \\ \tilde{\theta}(t) &= \end{aligned}$$

- (h) (4 pts.) Make a **rough** sketch of your linearized solutions for $y(t)$ and $\theta(t)$ for $0 \leq t \leq 16$ sec.



- (i) (6 pts.) The computer solution for \tilde{y} and $\tilde{\theta}$ to the original nonlinear ODEs for $0 \leq t \leq 16$ sec are plotted below. Comment on the similarities and differences between the plots you produced with the ones below. Give a reason for the similarities and differences.



Similarities:

Differences:

Reason for differences:

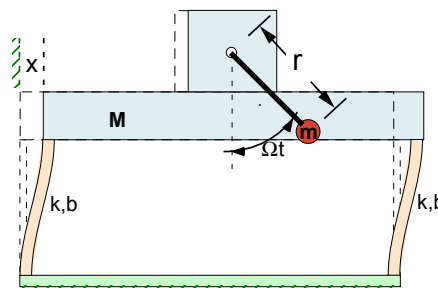
Similarities:

Differences:

Reason for differences:

F.4 (29 pts.) Dynamic response for an air-conditioner on a building

An air conditioner is bolted to the roof of a one story building. The air conditioner's motor is unbalanced and its eccentricity is modeled as a particle of mass m attached to the distal end of a rigid rod of length r . When the motor spins with angular speed Ω , it causes the building's roof of mass M to vibrate. The stiffness and material damping in *each* column that supports the roof is modeled as a linear horizontal spring (k) and linear horizontal damper (b). Homework 4.5 showed that the ODE governing the horizontal displacement x of the building's roof is



$$(M+m)\ddot{x} + 2b\dot{x} + 2kx = mr\Omega^2 \sin(\Omega t)$$

- (a) **(3 pts.)** Express ω_n , ζ , A in terms of M , m , b , k , r so the ODE is

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2x = A\Omega^2 \sin(\Omega t)$$

Result:

$$\omega_n = \qquad \qquad \zeta = \qquad \qquad A =$$

- (b) **(2 pts.)** For certain values of M , m , b , k , and r , this ODE simplifies to

$$\ddot{x} + 3\dot{x} + 900x = 1 \times 10^{-4} \Omega^2 \sin(\Omega t)$$

Calculate numerical values for ω_n and ζ .

Result:

$$\omega_n = \frac{\text{rad}}{\text{sec}} \qquad \zeta =$$

- (c) **(7 pts.)** Fill in numerical values in the following expressions for $x_{\text{ss}}(t)$, the steady-state part of $x(t)$.

Ω ($\frac{\text{rad}}{\text{sec}}$)	$x_{\text{ss}}(t)$
20	* $\sin(\quad t + \quad)$
30	* $\sin(\quad t + \quad)$
40	* $\sin(\quad t + \quad)$

- (d) **(14 pts.)** The building's occupants complain that the roof shakes too much. Comment on the effect small variations of M , m , b , k , and r have on: ζ , $\frac{\Omega}{\omega_n}$, and the magnitude of $x_{ss}(t)$. Fill in each element in the table by writing $-$ (decreases), $\mathbf{0}$ (no effect), $+$ (increases), or $?$ (if it may decrease *or* increase). For the 2nd-to-last column, assume the air conditioner's normal operating speed is $\Omega = 28 \frac{\text{rad}}{\text{sec}}$, and for last column, assume $\Omega = 32 \frac{\text{rad}}{\text{sec}}$.

	ζ	$\frac{\Omega}{\omega_n}$	$\Omega \approx 28 \frac{\text{rad}}{\text{sec}}$ $ x_{ss}(t) $	$\Omega \approx 32 \frac{\text{rad}}{\text{sec}}$ $ x_{ss}(t) $
Balancing the motor ($r \rightarrow 0$)	$\mathbf{0}$	$\mathbf{0}$	—	
Increasing the motor speed Ω				
Decreasing the motor speed Ω				
Adding mass to the roof (increasing M)				
Removing mass from the roof				
Stiffening the support columns (increasing k)				
Adding damping to the columns (increasing b)				

- (e) **(3 pts.)** List three ways to change the motor and minimize the roof shaking.