Math 51 - Winter 2011 - Final

Name: ______________________________

Student ID: ______________________________

Circle your section meting time:

Nick Haber 11:00 AM 1:15 PM	James Zhao 10:00 AM 1:15 PM	Henry Adams 11:00 AM 1:15 PM
Ralph Furmaniak 11:00 AM 1:15 PM	Jeremy Miller 11:00 AM 2:15 PM	Ha Pham 11:00 AM 1:15 PM
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Signature: ______________________________

Instructions: Print your name and student ID number, select the time at which your section meets, and **write your signature to indicate that you accept the Honor Code**. There are 15 problems on the pages numbered from 1 to 16. Each problem is worth 10 points. In problems with multiple parts, the parts are worth an equal number of points unless otherwise noted. Please check that the version of the exam you have is complete, and correctly stapled. In order to receive full credit, please show all of your work and justify your answers. You may use any result from class, but if you cite a theorem be sure to verify the hypotheses are satisfied. **You have 3 hours.** This is a closed-book, closed-notes exam. No calculators or other electronic aids will be permitted. GOOD LUCK!

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	Total

1(a). Let $f(x,y,z) = 3y^2 + 2y^3 - 3x^2 + 6xy + z^2$. Find the second order Taylor approximation of f at point $(0,-1,1)$.

Solution: We use Definition 11.3 from the DVC book, which says

$$T_2(\mathbf{x}) = f(\mathbf{a}) + D_f(\mathbf{a})(\mathbf{x}-\mathbf{a}) + \frac{1}{2}(\mathbf{x}-\mathbf{a})^T H_f(\mathbf{a})(\mathbf{x}-\mathbf{a}).$$

Here, $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\mathbf{a} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$.

We calculate $f(\mathbf{a}) = 2$ and

$$D_f(\mathbf{a}) = \begin{bmatrix} -6x+6y & 6y+6y^2+6x & 2z \end{bmatrix}_{(0,-1,1)} = \begin{bmatrix} -6 & 0 & 2 \end{bmatrix}$$

and

$$H_f(\mathbf{a}) = \begin{bmatrix} -6 & 6 & 0 \\ 6 & 6+12y & 0 \\ 0 & 0 & 2 \end{bmatrix}_{(0,-1,1)} = \begin{bmatrix} -6 & 6 & 0 \\ 6 & -6 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Hence

$$T_2(\mathbf{x}) = 2 + \begin{bmatrix} -6 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y+1 \\ z-1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} x & y+1 & z-1 \end{bmatrix} \begin{bmatrix} -6 & 6 & 0 \\ 6 & -6 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y+1 \\ z-1 \end{bmatrix}.$$

It is totally fine (in fact preferred!) to not expand this expression any further.

1(b). Let S be the surface defined by $3y^2 + 2y^3 - 3x^2 + 6xy + z^2 = 2$. Find an equation for the tangent plane to S at $(0,-1,1)$.

Solution: Let $F : \mathbf{R}^3 \to \mathbf{R}$ be defined by $F(x,y,z) = 3y^2 + 2y^3 - 3x^2 + 6xy + z^2$. Note that $S = F^{-1}(2) = F^{-1}(F(0,-1,1))$. Hence Theorem 18 from the DVC book tells us that the tangent plane to S at $(0,-1,1)$ is given by the equation

$$0 = F_x(0,-1,1)(x-0) + F_y(0,-1,1)(y+1) + F_z(0,-1,1)(z-1).$$

As in part (a), we calculate that $F_x(0,-1,1) = -6$, $F_y(0,-1,1) = 0$ and $F_z(0,-1,1) = 2$. Hence the equation for the tangent plane is

$$0 = -6(x-0) + 0(y+1) + 2(z-1)$$

or

$$0 = -6x + 2(z-1).$$

2. Suppose the temperature at point (x, y) is $f(x, y) = y^2 - 4y + x^2 - 1$.

2(a). Find the hottest point(s) and the coldest point(s) on the ellipse

$$2x^2 + y^2 = 9.$$

Solution: $\nabla f = \lambda \nabla g$ implies $2x = 4\lambda x$ and $2y - 4 = 2\lambda y$. Then eliminating λ gives $x(y - 4) = 0$. $y = 4$ is not possible on the ellipse. So $x = 0$, thus $y^2 = 9$. This means $y = 3$ or $y = -3$. Thus, the two critical points are $(0, 3)$ and $(0, -3)$.
$f(0, 3) = -4$, and $f(0, -3) = 20$.
The hottest point is at $(0, -3)$ and the coldest point is at $(0, 3)$.

2(b). Find the hottest point(s) and the coldest point(s) on the region

$$2x^2 + y^2 \leq 9.$$

Solution: Solve

$$\nabla f(x) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2x \\ 2y - 4 \end{bmatrix}$$

There is one critical point at (0,2), which is within the domain. Check that this critical point is a local minimum (with Hessian matrix). Then compare: $f(0, 2) = -5 < f(0, 3) = -4$. And $f(0, -3) = 20$. The hottest point is still at $(0, -3)$ and the new coldest point is at $(0, 2)$.

3. Find all solutions to the following system of equations:

$$\begin{aligned} x_1 + 2x_2 + x_3 + x_4 &= 7 \\ x_1 + 2x_2 + 2x_3 - x_4 &= 12 \\ 2x_1 + 4x_2 + 6x_4 &= 4. \end{aligned}$$

Solution: Augmented matrix:

$$\begin{bmatrix} 1 & 2 & 1 & 1 & 7 \\ 1 & 2 & 2 & -1 & 12 \\ 2 & 4 & 0 & 6 & 4 \end{bmatrix}$$

Reduced row echelon form:

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 2 \\ 0 & 0 & 1 & -2 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

There are infinitely many solutions verifying:

$$x_1 + 2x_2 + 3x_4 = 2 \text{ and } x_3 - 2x_4 = 5$$

That is:

$$\left\{ s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 5 \\ 0 \end{bmatrix}, (s,t) \in \mathbf{R}^2 \right\}$$

4(a). Let $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ be vectors in $\mathbf{R}^4$. Prove that there is a nonzero vector $\mathbf{x}$ that is perpendicular to each of those vectors.

Solution: The problem asks us to find a non-zero $\mathbf{x}$ so that $\mathbf{v}_1 \cdot \mathbf{x} = \mathbf{v}_2 \cdot \mathbf{x} = \mathbf{v}_3 \cdot \mathbf{x} = 0$. This is equivalent to finding a non-zero element in the null space of the matrix

$$A = \begin{bmatrix} - & \mathbf{v}_1^T & - \\ - & \mathbf{v}_2^T & - \\ - & \mathbf{v}_3^T & - \end{bmatrix},$$

This is a 3×4 matrix (the $\mathbf{v}_i$ are in $\mathbb{R}^4$), so by the rank nullity theorem, null $= 4 -$ rank. Since rank ≤ 3, null ≥ 1, i.e. there is at least one non-zero vector in the the null space of A.

4(b). Suppose that $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are nonzero vectors that are orthogonal to each other. Prove that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

Solution: Suppose that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}.$$

To show that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent, we must show that $c_1 = c_2 = c_3 = 0$. Keep in mind that the problem asks us to assume that the vectors are non-zero and mutually orthogonal. Taking the dot product of both sides of this equation with $\mathbf{v}_1$, we get

$$0 = \mathbf{v}_1 \cdot (c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3) = c_1\mathbf{v}_1 \cdot \mathbf{v}_1 + c_2\mathbf{v}_1 \cdot \mathbf{v}_2 + c_3\mathbf{v}_1 \cdot \mathbf{v}_3.$$

The second two terms on the right hand side are zero by orthogonality, so we get

$$0 = c_1\mathbf{v}_1 \cdot \mathbf{v}_1.$$

Many students reached this step and concluded that $c_1 = 0$ without justification. This is indeed true, but only because we are assuming that $\mathbf{v}_1 \neq 0$, and therefore $\mathbf{v}_1 \cdot \mathbf{v}_1$, the square norm of $\mathbf{v}_1$, is not zero.

The same procedure, but this time using $\mathbf{v}_2$ and $\mathbf{v}_3$, shows that c_2 and c_3 are also zero.

5. Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a basis for $\mathbf{R}^3$, and suppose $T : \mathbf{R}^3 \to \mathbf{R}^3$ is a linear transformation such that

$$T(\mathbf{v}_1) = 7\mathbf{v}_3 \qquad T(\mathbf{v}_2) = \mathbf{v}_1 \qquad T(\mathbf{v}_3) = 9\mathbf{v}_2.$$

5(a). Find the matrix B for T with respect to the basis $\mathcal{B}$.

Solution: Recall that the matrix B for T with respect to the basis $\mathcal{B}$ is the matrix B such that

$$[T(\mathbf{v})]_\beta = B\,[\mathbf{v}]_\beta\,,$$

for all $\mathbf{v} \in \mathbf{R}^3$. So taking $\mathbf{v} = \mathbf{v}_1$ gives that

$$7\mathbf{e}_3 = [7\mathbf{v}_3]_\beta = [T(\mathbf{v}_1)]_\beta = B\,[\mathbf{v}_1]_\beta = B\mathbf{e}_1.$$

So the first column of B is given by 7 times $\mathbf{e}_3$. Taking $\mathbf{v} = \mathbf{v}_2$ and $\mathbf{v} = \mathbf{v}_3$ one sees that

$$B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 9 \\ 7 & 0 & 0 \end{pmatrix}.$$

5(b). Suppose that

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}.$$

Find the matrix A for T with respect to the standard basis for $\mathbf{R}^3$.

[Hint: You may use your answer to **5(a)**. However, it is easier to find A directly, without using the matrix B.]

Solution: We now want to find the matrix A for T relative to the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ are there are two ways we can do this. One is that we can relate A and B (from part a)) using the change of basis matrix but we prefer not to do this since we will need to calculate an inverse. Recall that the first column of A is given by $T(\mathbf{e}_1)$, second column by $T(\mathbf{e}_2)$ and third column by $T(\mathbf{e}_3)$. But note that $\mathbf{v}_1 = \mathbf{e}_1$ and so that

$$T(\mathbf{e}_1) = T(\mathbf{v}_1) = 7\mathbf{v}_3 = [7 \;\; 7 \;\; 21]^T.$$

Now $\mathbf{v}_2 = \mathbf{e}_1 + \mathbf{e}_2$ and so we have that

$$\mathbf{e}_2 = \mathbf{v}_2 - \mathbf{e}_1$$

and so that

$$T(\mathbf{e}_2) = T(\mathbf{v}_2) - T(\mathbf{e}_1) = \mathbf{v}_1 - 7\mathbf{v}_3 = [-6 \;\; -7 \;\; -21]^T.$$

Note that

$$\mathbf{v}_3 = \mathbf{v}_2 + 3\mathbf{e}_3$$

and so

$$3T(\mathbf{e}_3) = T(\mathbf{v}_3) - T(\mathbf{v}_2) = 9\mathbf{v}_2 - \mathbf{v}_1 = [8 \;\; 9 \;\; 0]^T,$$

so

$$A = \begin{pmatrix} 7 & -6 & \frac{8}{3} \\ 7 & -7 & 3 \\ 21 & -21 & 0 \end{pmatrix}.$$

6. Find A^{-1}, where $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

Solution: We need to row reduce the matrix

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

so the matrix on the left is in row reduced echelon form. The first step is to put the second row up top and the third row just below. The top row becomes the third row and so we have

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{array}\right).$$

To row reduce this we multiply the first row by -1 and add this to the third row and then multiply the second row by -1 and add this to the third row to get

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{array}\right)$$

and so

$$A^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & -1 \end{pmatrix}.$$

7. Find the area of the triangle with vertices $(1, 0, -1)$, $(-2, 1, 1)$, and $(0, 0, 0)$.

Solution: Let $\mathbf{u} = (1, 0, -1)$, $\mathbf{v} = (-2, 1, 1)$ and θ be the angle between the two vectors. The formula for the area Δ of the triangle is

$$\Delta = \frac{1}{2}||\mathbf{u}|| \cdot ||\mathbf{v}|| \sin(\theta)$$

Consider $||\mathbf{u} \times \mathbf{v}|| = ||\mathbf{u}|| \cdot ||\mathbf{v}|| \sin(\theta)$. Thus $\Delta = ||\mathbf{u} \times \mathbf{v}||/2$. Now we calculate $\mathbf{u} \times \mathbf{v} = (1, 1, 1)$ which has length $\sqrt{3}$, hence the area is $\sqrt{3}/2$.

An alternative: Use the dot product to work out $\cos(\theta)$ (but don't forget to simplify your final answer).

8. Let $f(x, y)$ be a scalar valued function of two variables describing the pressure at the point (x, y) on the (flat) Earth's surface. Suppose that

$$\frac{\partial f}{\partial x}(-1,2) = -1 \qquad \frac{\partial f}{\partial y}(-1,2) = 2.$$

8(a). Find the directional derivative of f at $(-1, 2)$ in the direction $\mathbf{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Solution: The directional derivative at $(-1, 2)$ in the direction of v is

$$\begin{aligned} \nabla f(-1,2)\frac{\mathbf{v}}{||\mathbf{v}||} &= \begin{bmatrix} \frac{\partial f}{\partial x}(-1,2) & \frac{\partial f}{\partial y}(-1,2) \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{2}} \\ &= \begin{bmatrix} -1 & 2 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \\ &= \frac{3}{\sqrt{2}} \end{aligned}$$

8(b). A dragon is flying along an isobar (i.e., a level set of f) with speed 500. At time $t = 0$, the insect is at the point $(-1, 2)$ and its x-coordinate is increasing. Find its velocity at time 0.

Solution: Let $\mathbf{v}$ be the dragonfly's velocity. Since it is flying along a level set, it must be perpendicular to the gradient:

$$0 = \mathbf{v} \cdot \nabla f(-1,2) = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \end{bmatrix} = -\mathbf{v}_1 + 2\mathbf{v}_2$$

Hence $\mathbf{v}_1 = 2\mathbf{v}_2$. We also know that the speed is 500, so:

$$\begin{aligned} 500 = ||v|| &= \sqrt{\mathbf{v}_1^2 + \mathbf{v}_2^2} \\ &= \sqrt{4\mathbf{v}_2^2 + \mathbf{v}_2^2} \\ &= \sqrt{5\mathbf{v}_2^2} \\ 100\sqrt{5} &= \sqrt{\mathbf{v}_2^2} = |\mathbf{v}_2| \end{aligned}$$

Thus either $\mathbf{v}_2 = 100\sqrt{5}$ and $\mathbf{v}_1 = 2\mathbf{v}_2 = 200\sqrt{5}$, or $\mathbf{v}_2 = -100\sqrt{5}$ and $\mathbf{v}_1 = 2\mathbf{v}_2 = -200\sqrt{5}$. The latter is impossible since we are told that the x-coordinate is increasing. Hence the velocity must be

$$\mathbf{v} = \begin{bmatrix} 200\sqrt{5} \\ 100\sqrt{5} \end{bmatrix}$$

9. Our dragon friend has now quit flying and is trying to build a box out of plywood. He has $12m^2$ of plywood available, and is building a box in the shape of a rectangular prism without a top (so just needs to create four sides and a bottom). What is the greatest volume that this box can contain? (to be precise, suppose he has found a flat lid elsewhere).

Solution: We are trying to maximize the function $V(x, y, z) = xyz$, subject to the constraint $S(x, y, z) = xy + 2xz + 2yz \leq 12$. Clearly the largest volume will use all of the plywood availiable, so we can assume $S(x, y, z) = 12$. Also, we can assume that x, y and z are all positive numbers, otherwise the equations have no physical interpretation. We use Lagrange multipliers.

$\nabla V = (yz, xz, xy)$ and $\nabla S = (y + 2z, x + 2z, 2x + 2y)$. Setting up the equation $\nabla V = \lambda \nabla S$, we get three equations,

$$yz = \lambda(y + 2z)$$
$$xz = \lambda(x + 2z)$$
$$xy = \lambda(2x + 2y).$$

Solving each equation here for λ, we get one big equation

$$\frac{yz}{y + 2z} = \frac{xz}{x + 2z} = \frac{xy}{2x + 2y}.$$

Notice that all our divisions are legal since our variables are positive. Cross multiplying fractions in the first equation gives $yz(x + 2z) = xz(y + 2z)$, which reduces to $2yz^2 = 2xz^2$. Since $z \neq 0$, this implies $x = y$.

By cross multiplying our second equation, we get $xz(2x+2y) = xy(x+2z)$. By reducing, we get $2x^2z = x^2y$, which implies $2z = y$.

Now that we know $x = y = 2z$, we can plug everything back into our formula $S(x, y, z) = 12$, which says $4z^2 + 4z^2 + 4z^2 = 12$. This tells us that $z = 1$, so $x = y = 2$. This gives a volume of 4, which must be the maximum, since there are no other critical points.

Some students started with the assumption that $x = y$, this is valid, because x and y are symmetric in the problem. Many students started with the assumption $x = y = z$, which does not work. The problem is not symmetrical this way, because of the open top.

10. Consider the matrices

$$A = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 2 & 1 & 4 & -1 \\ 1 & 0 & 2 & 1 \end{bmatrix} \quad , \quad R = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The matrix R is the row reduced echelon form of A (You do not need to check this).

10(a). Find a basis for the column space $C(A)$ of A.

Solution. In the row reduced echelon form R, the two leftmost columns are the ones that have been cleared out. Hence, the corresponding columns of the original matrix A are a basis for the column space $C(A)$. In other words, an answer to part (a) is

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

10(b). Find a basis for the null space $N(R)$ of R.

Solution: Using the matrix R, we see that the original system of equations is equivalent to the simplified system

$$x_1 = -2x_3 - x_4, \;\; x_2 = 3x_4.$$

Thus, the null space consists of exactly the vectors

$$\begin{bmatrix} -2x_3 - x_4 \\ 3x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 3 \\ 0 \\ 1 \end{bmatrix}.$$

Hence, an answer to part (b) is

$$\left\{ \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

10(c). Note that $A\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}4\\6\\4\end{bmatrix}$. Find all solutions to $A\mathbf{x} = \begin{bmatrix}4\\6\\4\end{bmatrix}$.

Solution: Finally, for part (c), we are given an explicit solution to the system of equations $A\mathbf{x} = \begin{bmatrix}4\\6\\4\end{bmatrix}$, namely $\mathbf{x}_0 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$. Thus, the general solution is given by the special solution $\mathbf{x}_0$ plus an element of the null space. In part (b) we found a basis for the null space, so we conclude that a general solution is given by

$$\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + s\begin{bmatrix}-2\\0\\1\\0\end{bmatrix} + t\begin{bmatrix}-1\\3\\0\\1\end{bmatrix}$$

with $s, t \in \mathbf{R}$. That is an answer to part (c).

(A notable feature of many valid solutions to part (c) different from the above is that a lot of people ignored the given information of a special solution $\mathbf{x}_0$ and instead "re-invented the wheel" by grinding through the usual manipulations to discover their own special solution. The most popular one found was $\begin{bmatrix}4\\0\\-2\\0\end{bmatrix}$. That is a valid and received full marks, but it involves more work than is really necessary: as we saw above, all of the necessary work has already been done in part(b)!)

Typical errors for solutions to problem 10. A typical error when solving part (b) was to either lose some signs, or get the entries mixed up in the vertical direction. One should *always* double check that the proposed basis for the null space at least does satisfy the original homogeneous system of linear equations corresponding to the given matrix A! That will catch most computational errors.

Another typical error for solutions to part (b) was to not give a basis, but rather to say that the null space consists of vectors of the general parametric form in the displayed equation above. That is not a valid answer (led to a 1-point deduction) because a *basis* of the null space is not the same thing as the null space itself. It is important to understand the distinction between these notions, even though passing between them is rather straightforward.

The typical error for solutions to part (c) was to "add" elements of the null space to $\begin{bmatrix} 4 \\ 6 \\ 4 \end{bmatrix}$ rather than to $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, but it is meaningless to "add" vectors with different numbers of entries (let alone to overlook the fact that one has to actually do the translation by a solution to the inhomogeneous system of equations). So that is a serious conceptual error.

11. Let $f : \mathbf{R}^2 \to \mathbf{R}$ be defined by $f(\mathbf{x}) = ||\mathbf{x}||$.

11(a). Compute $D_f(1,0)$.

Solution:

$$\begin{aligned}
f(x,y) &= (x^2+y^2)^{1/2} \\
\frac{\partial f}{\partial x} &= x(x^2+y^2)^{-1/2} \\
\frac{\partial f}{\partial y} &= y(x^2+y^2)^{-1/2} \\
D_f(x,y) &= \begin{bmatrix} x(x^2+y^2)^{-1/2} & y(x^2+y^2)^{-1/2} \end{bmatrix} \\
D_f(1,0) &= \begin{bmatrix} 1 & 0 \end{bmatrix}.
\end{aligned}$$

Common mistakes were confusing total derivatives and gradients, thinking this was a function of one variable, or forgetting the chain rule.

11(b). Show that f is not differentiable at $(0,0)$.

Solution:

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h)-f(0)}{h-0} = \lim_{h\to 0} |h|/h$$

which does not exist since $\lim_{h\to 0^+} |h|/h = 1$ and $\lim_{h\to 0^-} |h|/h = -1$. Differentiable functions have all of their partial derivatives so f is not differentiable.

Common mistakes were thinking one could prove a function was not differentiable by proving its derivative is not continuous, thinking that the square root of zero is indeterminate and forgetting that sometimes zero over zero issues can be resolved by LH's rule.

12. If $t = x^2 + yz^2$ and $x = uve^{2s}$, $y = u^2 - v^2 s$, $z = \cos(uvs)$.

12(a). Find $\frac{\partial t}{\partial u}(2, 1, 0)$.

12(b). Find $\frac{\partial t}{\partial s}(0, 1, 5)$.

Solution. There are several ways to solve this. A widely-used method was brute force: explicitly plug the expressions for x, y, and z in terms of u, v, and s into the definition of t to get

$$t = u^2 v^2 e^{4s} + (u^2 - v^2 s)\cos(uvs)^2$$

(some erred here by not remembering the laws of exponents), and then explicitly differentiate this with respect to u and s separately. This leads to rather elaborate expressions (which are then evaluated at $(2, 1, 0)$ for (a) and at $(0, 1, 5)$ for (b)). Namely, one has

$$\frac{\partial t}{\partial u} = 2uv^2 e^{4s} + 2u\cos(uvs)^2 - 2vs(u^2 - v^2 s)\cos(uvs)\sin(uvs),$$
$$\frac{\partial t}{\partial s} = 4u^2 v^2 e^{4s} - v^2\cos(uvs)^2 - 2uv(u^2 - v^2)\cos(uvs)\sin(uvs),$$

which respectively evaluate to 8 and -1 at the indicated points in parts (a) and (b). This is a very painful way to arrive at the answer.

The method which was intended for the solution was to use the Chain Rule (to ease the algebra burden):

$$\frac{\partial t}{\partial u} = \frac{\partial t}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial t}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial t}{\partial z}\cdot\frac{\partial z}{\partial u}$$

and similarly for $\frac{\partial t}{\partial s}$ by replacing u with s everywhere. In this approach, one has to keep a clear distinction between the point (u_0, v_0, s_0) at which we wish to evaluate the left side (namely, (2,1,0) for part (a) and (0,1,5) for part (b)) and the corresponding point in "(x, y, z) coordinates":

$$(x(2, 1, 0), y(2, 1, 0), z(2, 1, 0)) = (2, 4, 1),$$
$$(x(0, 1, 5), y(0, 1, 5), z(0, 1, 5)) = (0, -5, 1).$$

To be precise, the "Chain Rule" expression for $\frac{\partial t}{\partial u}$ evaluated at $(2, 1, 0)$ simplifies to give

$$\frac{\partial t}{\partial u}(2, 1, 0) = (2x + 4z^2)|_{(2,4,1)} = 8,$$

and proceeding similarly for the s-partial gives

$$\frac{\partial t}{\partial s}(0, 1, 5) = -z^2|_{(0,-5,1)} = -1.$$

There is a third method of solution which truly minimizes the odds of algebra errors by going back to the very definition of a partial derivative and avoiding the Chain Rule altogether: when computing a u-partial at the point $(2,1,0)$, it doesn't matter whether we replace (v,s) with $(1,0)$ before *or* after the derivative calculation with respect to u. (Why not? Think about it.) So let's do that evaluation *before* the differentiation: $t(u,1,0) = u^2 + u^2 \cdot 1 = 2u^2$. The u-derivative of this is $4u$, and setting $u = 2$ then recovers the answer of 8 for part (a). Likewise, for part (b) we observe that $t(0,1,s) = -s\cos(0)^2 = -s$, so its s-derivative is -1 (a constant!). Evaluating at $s = 5$ recovers the answer -1 for part (b).

Typical errors for solutions to problem 12. In solutions to by the brute force method, apart from the many varieties of algebra errors the most typical error by far was to overlook the minus sign in front the v^2 in the middle term of the "brute force" expression for $\frac{\partial t}{\partial s}$ (leading to an answer of 1 rather than -1 for part (b)). In the solutions via the Chain Rule method, the typical error was to forget that when working in terms of (x,y,z) we need to recompute the point we're at (namely, $(2,4,1)$ for part (a), and $(0,-5,1)$ for part (b)). Also, a lot of people mistakenly evaluated the expression $\frac{\partial t}{\partial x} = 2x$ at the point $(2,4,1)$ to be 2 rather than $2 \cdot 2 = 4$.

13(a). Let $f(x,y) = 3x + Ax^3 + Bxy^2$ for some constants A and B. Find A and B if it is known that the function f has critical points at $(1,0)$ and $(0,1)$.

Solution: Recall that f has a critical point at (x_0, y_0) if and only if

$$\nabla f(x_0, y_0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Computing the gradient we have

$$\nabla f(x,y) = \begin{bmatrix} 3 + 3Ax^2 + By^2 \\ 2xyB \end{bmatrix}$$

Thus

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \nabla f(1,0) = \begin{bmatrix} 3 + 3A \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \nabla f(0,1) = \begin{bmatrix} 3 + B \\ 0 \end{bmatrix}$$

which implies $A = -1$ and $B = -3$.

13(b). Determine for each of the two points $(1,0)$ and $(0,1)$ if it is a local maximum, a local minimum, or a saddle point.

Solution: All this information can be recovered from the Hessian of f. Taking the appropriate derivatives and plugging in the values for A, B we get:

$$Hess\ f(x,y) = \begin{bmatrix} -6x & -6y \\ -6y & -6x \end{bmatrix}$$

Therefore

$$Hess\ f(1,0) = \begin{bmatrix} -6 & 0 \\ 0 & -6 \end{bmatrix} \quad \text{and} \quad Hess\ f(0,1) = \begin{bmatrix} 0 & -6 \\ -6 & 0 \end{bmatrix}$$

- $det(Hess\ f(1,0)) = 36 > 0$, $trace(Hess\ f(0,1)) = -12 < 0$. Thus, both eigenvalues of $Hess\ f(1,0)$ are negative and $(1,0)$ is a local maximum.
- $det(Hess\ f(0,1)) = -36 < 0$ Thus, the eigenvalues of $Hess\ f(0,1)$ have mixed signs and therefore $(0,1)$ is a saddle point.

14. Suppose $F : \mathbf{R}^2 \to \mathbf{R}^3$ is defined by $F(x,y) = \begin{bmatrix} e^{e^{xy}} \\ e^{\sin x} - y^2 \\ e^{\cos y} + x^2 \end{bmatrix}$.

Find the Jacobian matrix (i.e, the total derivative matrix) $D_F(x,y)$.

Solution: Recall that the Jacobian of a function $F : \mathbf{R}^2 \longrightarrow \mathbf{R}^3$ is a 3×2 matrix given by the formula:

$$D_F(x,y) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} \end{bmatrix} \quad \text{where} \quad F(x,y) = \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \\ f_3(x,y) \end{bmatrix}$$

We proceed now to computing the relevant derivatives. To make the use of the chain rule more transparent, we let $f(t) = e^t$. It follows that

$$\begin{aligned} \frac{\partial}{\partial x} e^{e^{xy}} &= \frac{\partial}{\partial x} f(f(xy)) = \frac{\partial f}{\partial t}(f(xy)) \cdot \frac{\partial f}{\partial t}(xy) \cdot \frac{\partial}{\partial x}(xy) \\ &= e^{f(xy)} \cdot e^{xy} \cdot y \\ &= y e^{e^{xy} + xy} \end{aligned}$$

Following this procedure, we get the matrix of derivatives:

$$D_F(x,y) = \begin{bmatrix} y e^{e^{xy}+xy} & x e^{e^{xy}+xy} \\ \cos x \cdot e^{\sin x} & -2y \\ 2x & -\sin y \cdot e^{\cos y} \end{bmatrix}$$

15(a). Find an eigenvector with eigenvalue $\lambda = 1$ for the matrix

$$A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

Solution: Since each row in this matrix adds up to 1, it follows that $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector of A with eigenvalue 1. Another way of going about this, is to actually compute the eigenspace associated to λ_1. That is,

$$E_1 = N(I - A) = N\left(\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}\right)$$

Putting this new matrix in RREF we get

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

and therefore

$$E_1 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \;\middle|\; x = y = z \right\}$$

15(b). Find an eigenvector with eigenvalue $\lambda = 1$ for the matrix $A^2 + A - I$, where I is the identity matrix and A is the matrix in part (a).

Solution: The thing to notice here is that if $\mathbf{v}$ is an eigenvector of A with eigenvalue 1, then

$$\begin{aligned} (A^2 - A + I)\mathbf{v} &= A(A\mathbf{v}) - A\mathbf{v} + I\mathbf{v} \\ &= A\mathbf{v} - \mathbf{v} + \mathbf{v} \\ &= \mathbf{v} \end{aligned}$$

and therefore $\mathbf{v}$ is also an eigenvector of $A^2 - A + I$ with eigenvalue 1. That is, any answer for part (a), will also be an answer for part (b).