Math 51 - Winter 2011 - Final

Name: ______________________________

Student ID: ______________________________

Circle your section meting time:

Nick Haber 11:00 AM 1:15 PM	James Zhao 10:00 AM 1:15 PM	Henry Adams 11:00 AM 1:15 PM
Ralph Furmaniak 11:00 AM 1:15 PM	Jeremy Miller 11:00 AM 2:15 PM	Ha Pham 11:00 AM 1:15 PM
Sukhada Fadnavis 10:00 AM 1:15 PM	Max Murphy 11:00 AM 1:15 PM	Jesse Gell-Redman 1:15 PM

Signature: ______________________________

Instructions: Print your name and student ID number, select the time at which your section meets, and **write your signature to indicate that you accept the Honor Code**. There are 15 problems on the pages numbered from 1 to 16. Each problem is worth 10 points. In problems with multiple parts, the parts are worth an equal number of points unless otherwise noted. Please check that the version of the exam you have is complete, and correctly stapled. In order to receive full credit, please show all of your work and justify your answers. You may use any result from class, but if you cite a theorem be sure to verify the hypotheses are satisfied. **You have 3 hours.** This is a closed-book, closed-notes exam. No calculators or other electronic aids will be permitted. GOOD LUCK!

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	Total

1(a). Let $f(x, y, z) = 3y^2 + 2y^3 - 3x^2 + 6xy + z^2$. Find the second order Taylor approximation of f at point $(0, -1, 1)$.

1(b). Let S be the surface defined by $3y^2 + 2y^3 - 3x^2 + 6xy + z^2 = 2$. Find an equation for the tangent plane to S at $(0, -1, 1)$.

2. Suppose the temperature at point (x, y) is $f(x, y) = y^2 - 4y + x^2 - 1$.

2(a). Find the hottest point(s) and the coldest point(s) on the ellipse

$$2x^2 + y^2 = 9.$$

2(b). Find the hottest point(s) and the coldest point(s) on the region

$$2x^2 + y^2 \leq 9.$$

3. Find all solutions to the following system of equations:

$$
\begin{aligned}
x_1 + 2x_2 + x_3 + x_4 &= 7 \\
x_1 + 2x_2 + 2x_3 - x_4 &= 12 \\
2x_1 + 4x_2 + 6x_4 &= 4.
\end{aligned}
$$

4(a). Let $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ be vectors in $\mathbf{R}^4$. Prove that there is a nonzero vector $\mathbf{x}$ that is perpendicular to each of those vectors.

4(b). Suppose that $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are nonzero vectors that are orthogonal to each other. Prove that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

5. Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a basis for $\mathbf{R}^3$, and suppose $T : \mathbf{R}^3 \to \mathbf{R}^3$ is a linear transformation such that

$$T(\mathbf{v}_1) = 7\mathbf{v}_3 \qquad T(\mathbf{v}_2) = \mathbf{v}_1 \qquad T(\mathbf{v}_3) = 9\mathbf{v}_2.$$

5(a). Find the matrix B for T with respect to the basis $\mathcal{B}$.

5(b). Suppose that

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}.$$

Find the matrix A for T with respect to the standard basis for $\mathbf{R}^3$.

[Hint: You may use your answer to **5(a)**. However, it is easier to find A directly, without using the matrix B.]

6. Find A^{-1}, where $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

7. Find the area of the triangle with vertices $(1, 0, -1)$, $(-2, 1, 1)$, and $(0, 0, 0)$.

8. Let $f(x, y)$ be a scalar valued function of two variables describing the pressure at the point (x, y) on the (flat) Earth's surface. Suppose that

$$\frac{\partial f}{\partial x}(-1, 2) = -1 \qquad \frac{\partial f}{\partial y}(-1, 2) = 2.$$

8(a). Find the directional derivative of f at $(-1, 2)$ in the direction $\mathbf{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

8(b). A dragon is flying along an isobar (i.e., a level set of f) with speed 500. At time $t = 0$, the insect is at the point $(-1, 2)$ and its x-coordinate is increasing. Find its velocity at time 0.

9. Our dragon friend has now quit flying and is trying to build a box out of plywood. He has $12m^2$ of plywood available, and is building a box in the shape of a rectangular prism without a top (so just needs to create four sides and a bottom). What is the greatest volume that this box can contain? (to be precise, suppose he has found a flat lid elsewhere).

10. Consider the matrices

$$A = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 2 & 1 & 4 & -1 \\ 1 & 0 & 2 & 1 \end{bmatrix} \quad , \quad R = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The matrix R is the row reduced echelon form of A (You do not need to check this).

10(a). Find a basis for the column space $C(A)$ of A.

10(b). Find a basis for the null space $N(R)$ of R.

10(c). Note that $A \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 4 \end{bmatrix}$. Find all solutions to $A\mathbf{x} = \begin{bmatrix} 4 \\ 6 \\ 4 \end{bmatrix}$.

11. Let $f : \mathbf{R}^2 \to \mathbf{R}$ be defined by $f(\mathbf{x}) = ||\mathbf{x}||$.

11(a). Compute $D_f(1, 0)$.

11(b). Show that f is not differentiable at $(0, 0)$.

12. If $t = x^2 + yz^2$ and $x = uve^{2s}$, $y = u^2 - v^2 s$, $z = \cos(uvs)$.

12(a). Find $\frac{\partial t}{\partial u}(2, 1, 0)$.

12(b). Find $\frac{\partial t}{\partial s}(0, 1, 5)$.

13(a). Let $f(x, y) = 3x + Ax^3 + Bxy^2$ for some constants A and B. Find A and B if it is known that the function f has critical points at $(1, 0)$ and $(0, 1)$.

13(b). Determine for each of the two points $(1, 0)$ and $(0, 1)$ if it is a local maximum, a local minimum, or a saddle point.

14. Suppose $F : \mathbf{R}^2 \to \mathbf{R}^3$ is defined by $F(x,y) = \begin{bmatrix} e^{e^{xy}} \\ e^{\sin x} - y^2 \\ e^{\cos y} + x^2 \end{bmatrix}$.

Find the Jacobian matrix (i.e, the total derivative matrix) $D_F(x,y)$.

15(a). Find an eigenvector with eigenvalue $\lambda = 1$ for the matrix

$$A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

15(b). Find an eigenvector with eigenvalue $\lambda = 1$ for the matrix $A^2 + A - I$, where I is the identity matrix and A is the matrix in part (a).